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AN ANALOGUE OF SOME INEQUALITIES OF P. TURÁN CONCERNING ALGEBRAIC POLYNOMIALS HAVING ALL ZEROS INSIDE [− 1, + 1]. II

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(In memory of Professor P. Turán)

ABSTRACT. Let $P_n(x)$ be an algebraic polynomial of degree $< n$ having all zeros inside $[-1, +1]$; then we have

$$\int_{-1}^{1} P_n'^2(x)\,dx > \left(\frac{n}{2} + \frac{3}{4} + \frac{3}{4n}\right)\int_{-1}^{1} P_n^2(x)\,dx.$$

This bound is much sharper than found in [2]. Moreover, if $P_n(1) = P_n(-1) = 0$, then under the above conditions we have

$$\int_{-1}^{1} P_n'^2(x)\,dx > \left(\frac{n}{2} + \frac{3}{4} + \frac{3}{4(n-1)}\right)\int_{-1}^{1} P_n^2(x)\,dx,$$

equality for $P_n(x) = (1 - x^2)^m$, $n = 2m$.

1. Let H_n be the set of all polynomials whose degree is n and whose zeros are all real and lie inside $[-1, +1]$. Throughout this paper $x_1, x_2, \ldots, x_n$ will denote the roots of $P_n(x)$ ($P_n \in H_n$). In an earlier work [2] the following theorem was proved.

THEOREM A. *Let $P_n \in H_n$; then we have*

$$\begin{aligned} 2\int_{-1}^{1} P_n'^2(x)\,dx &> 2\int_{-1}^{1} (1 - x^2)P_n'^2(x)\,dx \\ &= n\int_{-1}^{1} P_n^2(x)\,dx + \int_{-1}^{1} P_n^2(x) \sum_{k=1}^{n} \frac{1 - x_k^2}{(x - x_k)^2}\,dx. \end{aligned} \tag{1.1}$$

Hence, for $P_n \in H_n$ we have

$$\|P_n'\|_{L_2[-1,+1]} > \frac{n^{1/2}}{2^{1/2}} \cdot \|P_n\|_{L_2[-1,+1]}. \tag{1.2}$$

($P_n'(x)$ stands for the derivative of $P_n(x)$).

(1.2) is analogous to P. Turán's [1] beautiful theorem in the L_∞ norm.

In this paper we give a more precise form of (1.2). We prove

THEOREM 1. *Let $P_n \in H_n$ and*

$$P_n(1) = P_n(-1) = 0; \tag{1.3}$$

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then we have

$$\|P_n'\|^2_{L_2[-1,+1]} \geq \left(\frac{n}{2}+\frac{3}{4}+\frac{3}{4(n-1)}\right)\|P_n\|^2_{L_2[-1,+1]} \tag{1.4}$$

with equality for $P_n(x) = (1-x^2)^m$, $n = 2m$.

THEOREM 2. *Let* $P_n \in H_n$; *then we have*

$$2\int_{-1}^{1} P_n'^2(x)\,dx > 2\int_{-1}^{1}(1-x^4)P_n'^2(x)\,dx \geq \alpha_{1,n}\int_{-1}^{1} P_n^2(x)\,dx, \tag{1.5}$$

$$2\int_{-1}^{1} P_n'^2(x)\,dx > 2\int_{-1}^{1}(1-x^6)P_n'^2(x)\,dx \geq \alpha_{2,n}\int_{-1}^{1} P_n^2(x)\,dx, \tag{1.6}$$

$$2\int_{-1}^{1} P_n'^2(x)\,dx > 2\int_{-1}^{1}(1-x^8)P_n'^2(x)\,dx \geq \alpha_{3,n}\int_{-1}^{1} P_n^2(x)\,dx, \tag{1.7}$$

where

$$\begin{aligned}\alpha_{1,n} &= n+\frac{3n}{2n+3}, \quad \alpha_{2,n} = \alpha_{1,n}+\frac{15n}{(2n+5)(2n+3)},\\ \alpha_{3,n} &= \alpha_{2,n}+\frac{105n}{(2n+7)(2n+5)(2n+3)}.\end{aligned} \tag{1.8}$$

Equality holds on the right-hand side of (1.5)–(1.7) *iff* $P_n(x) = (1-x^2)^m$, $n = 2m$. *From* (1.7) *we obtain* ($n \geq 3$)

$$\|P_n'\|^2_{L_2[-1,+1]} > \left(\frac{n}{2}+\frac{3}{4}+\frac{3}{4n}\right)\|P_n\|^2_{L_2[-1,+1]}. \tag{1.9}$$

This result is sharp for $P_{0n}(x) = (1-x^2)^m$, $n = 2m$; *indeed we have*

$$\|P_{0n}'\|^2_{L_2[-1,+1]} = \left(\frac{n}{2}+\frac{3}{4}+\frac{3}{4(n-1)}\right)\|P_{0n}\|^2_{L_2[-1,+1]}. \tag{1.10}$$

2. Some identities. For the proofs of Theorems 1 and 2 the following identities are needed.

IDENTITY 2.1. Let $P_n(x)$ be any polynomial of degree n; then for all positive integers r,

$$\begin{aligned}2\int_{-1}^{1}(1-x^{2r})P_n'^2(x)\,dx &= \sum_{i=1}^{r}(n+2i-1)\int_{-1}^{1}x^{2i-2}P_n^2(x)\,dx\\ &\quad - r(2r-1)\int_{-1}^{1}x^{2r-2}P_n^2(x)\,dx + \sum_{i=1}^{r}S_{2i-2},\end{aligned} \tag{2.1}$$

where

$$S_{2i} = \int_{-1}^{1}x^{2i}P_n^2(x)\sum_{k=1}^{n}\frac{1-x_k^2}{(x-x_k)^2}\,dx. \tag{2.2}$$

PROOF. On integrating by parts, we obtain

$$\int_{-1}^{1}(1 - x^{2r})P_n'(x)P_n'(x)\,dx$$

$$= -\int_{-1}^{1} P_n(x)\{(1 - x^{2r})P_n''(x) - 2rx^{2r-1}P_n'(x)\}\,dx;$$

we rewrite it as

$$2\int_{-1}^{1}(1 - x^{2r})P_n'^2(x)\,dx = \int_{-1}^{1}(1 - x^{2r})(P_n'^2(x) - P_n(x)P_n''(x))\,dx + 2r\int_{-1}^{1} x^{2r-1}P_n(x)P_n'(x)\,dx. \tag{2.3}$$

On using

$$P_n'^2(x) - P_n(x)P_n''(x) = P_n^2(x)\sum_{k=1}^{n}\frac{1}{(x - x_k)^2}, \tag{2.4}$$

$$\frac{(1 - x^2)}{(x - x_k)^2} = 1 - \frac{2x}{x - x_k} + \frac{1 - x_k^2}{(x - x_k)^2}, \tag{2.5}$$

and

$$P_n'(x) = P_n(x)\sum_{k=1}^{n}\frac{1}{(x - x_k)} \tag{2.6}$$

in (2.3) we obtain

$$\begin{aligned} 2\int_{-1}^{1}(1 - x^{2r})P_n'^2(x)\,dx &= 2r\int_{-1}^{1} x^{2r-1}P_n(x)P_n'(x)\,dx \\ &\quad + \int_{-1}^{1}\left(\sum_{i=1}^{r} x^{2i-2}\right)\left\{\sum_{k=1}^{n}\left[1 - \frac{2x}{x - x_k} + \frac{(1 - x_k^2)}{(x - x_k)^2}\right]P_n^2(x)\right\}dx \\ &= n\int_{-1}^{1}\left(\sum_{i=1}^{r} x^{2i-2}\right)P_n^2(x)\,dx + \sum_{i=1}^{r} S_{2i-2} \\ &\quad + 2\int_{-1}^{1}\left(rx^{2r-1} - \sum_{i=1}^{r} x^{2i-1}\right)P_n(x)P_n'(x)\,dx. \end{aligned} \tag{2.7}$$

On integration by parts we have

$$\begin{aligned} 2\int_{-1}^{1}&\left(rx^{2r-1} - \sum_{i=1}^{r} x^{2i-1}\right)P_n(x)P_n'(x)\,dx \\ &= -\int_{-1}^{1}\left\{(2r - 1)rx^{2r-2} - \sum_{i=1}^{r}(2i - 1)x^{2i-2}\right\}P_n^2(x)\,dx. \end{aligned} \tag{2.8}$$

Combining (2.7) and (2.8) we obtain (2.1).

IDENTITY 2.2. Let $P_n(x)$ be any polynomial of degree n, r any fixed positive integer; then we have

$$
\begin{aligned}
2\int_{-1}^{1} x^{2r-2}(1-x^2)^2 P_n'^2(x)\,dx = {}& (2r^2+r-n)\int_{-1}^{1} x^{2r}P_n^2(x)\,dx \\
&+ (n+4r-4r^2-1)\int_{-1}^{1} x^{2r-2}P_n^2(x)\,dx + S_{2r-2} - S_{2r} \\
&+ (r-1)(2r-3)\int_{-1}^{1} x^{2r-4}P_n^2(x)\,dx,
\end{aligned} \tag{2.9}
$$

where S_{2r} is defined by (2.2). For $r = 1$ the last term on the right-hand side at (2.9) does not appear.

The proof of this identity is similar to (2.1) and so we omit the details.

IDENTITY 2.3. Let $P_n(x)$ be any polynomial of degree n, r any fixed positive integer; then we have

$$
\begin{aligned}
(n+r)(2n+2r+1)\int_{-1}^{1} x^{2r}P_n^2(x)\,dx = {}& S_{2r} - S_{2r-2} \\
&+2\int_{-1}^{1} x^{2r-2}\big(nxP_n(x) + (1-x^2)P_n'(x)\big)^2\,dx \\
&+ (4nr+4r^2-4r-3n+1)\int_{-1}^{1} x^{2r-2}P_n^2(x)\,dx \\
&- (r-1)(2r-3)\int_{-1}^{1} x^{2r-4}P_n^2(x)\,dx.
\end{aligned} \tag{2.10}
$$

PROOF. We write

$$
\begin{aligned}
\int_{-1}^{1} x^{2r}P_n^2(x)\,dx = {}& \frac{1}{n^2}\int_{-1}^{1} x^{2r-2}\big(nxP_n(x) + (1-x^2)P_n'(x)\big)^2\,dx \\
&- \frac{1}{n^2}\int_{-1}^{1} x^{2r-2}(1-x^2)^2P_n'^2(x)\,dx \\
&- \frac{2}{n}\int_{-1}^{1} x^{2r-1}(1-x^2)P_n(x)P_n'(x)\,dx.
\end{aligned} \tag{2.11}
$$

On using (2.9) and

$$
\begin{aligned}
&\int_{-1}^{1} 2x^{2r-1}(1-x^2)P_n(x)P_n'(x)\,dx \\
&= -\int_{-1}^{1} \big((2r-1)x^{2r-2} - (2r+1)x^{2r}\big)P_n^2(x)\,dx,
\end{aligned} \tag{2.12}
$$

we obtain (2.10) on rearranging (2.11).

From (2.10) and

$$
\int_{-1}^{1} x^{2r}\big(nxP_n(x) + (1-x^2)P_n'(x)\big)^2\,dx \geqslant 0
$$

(equality for only $P_n(x) = (1-x^2)^m$, $n = 2m$, the following inequalities are obtained.

$$
\int_{-1}^{1} x^2P_n^2(x)\,dx \geqslant \frac{1}{2n+3}\int_{-1}^{1} P_n^2(x)\,dx + \frac{S_2 - S_0}{(n+1)(2n+3)}\,; \tag{2.13}
$$

$$\int_{-1}^{1} x^4 P_n^2(x)\,dx \geq \frac{3}{(2n+5)(2n+3)} \int_{-1}^{1} P_n^2(x)\,dx - \alpha_{4,n} S_0 - \alpha_{5,n} S_2 + \alpha_{6,n} S_4, \tag{2.14}$$

where

$$\alpha_{4,n} = \frac{5n+9}{(2n+5)(2n+3)(n+2)(n+1)}, \qquad \alpha_{5,n} = \frac{2n^2-6}{(2n+5)(2n+3)(n+2)(n+1)}, \tag{2.15}$$

$$\alpha_{6,n} = \frac{1}{(2n+5)(n+2)}. \tag{2.16}$$

$$\int_{-1}^{1} x^6 P_n^2(x)\,dx \geq \frac{15}{(2n+7)(2n+5)(2n+3)} \int_{-1}^{1} P_n^2(x)\,dx - \alpha_{7,n} S_0 - \alpha_{8,n} S_2 - \alpha_{9,n} S_4, \tag{2.17}$$

where

$$\alpha_{7,n} = \frac{33n^2 + 152n + 165}{(2n+7)(2n+5)(2n+3)(n+3)(n+2)(n+1)},$$
$$\alpha_{8,n} = \frac{18n^3 + 62n^2 - 90}{(2n+7)(2n+5)(2n+3)(n+3)(n+2)(n+1)}, \tag{2.18}$$
$$\alpha_{9,n} = \frac{2n^2 - 15}{(2n+7)(2n+5)(n+3)(n+2)}.$$

IDENTITY 2.4. Let $P_n(x)$ be any polynomial of degree n; then we have

$$2(n+3)\int_{-1}^{1} x^2 P_n^2(x)\,dx - (2n+5)\int_{-1}^{1} x^4 P_n^2(x)\,dx - \int_{-1}^{1} P_n^2(x)\,dx$$
$$= \frac{2}{(n+2)} \int_{-1}^{1} (1-x^2)\left(nxP_n(x) + (1-x^2)P_n'(x)\right)^2 dx \tag{2.19}$$
$$- \frac{1}{(n+2)} \int_{-1}^{1} (1-x^2)^2 \sum_{k=1}^{n} \frac{(1-x_k^2)}{(x-x_k)^2} P_n^2(x)\,dx.$$

PROOF. On using (2.3) for $r = 2$, and $r = 1$ on the left-hand side of (2.19) gives the result.

Replacing n in the above identity by $n-2$, and $P_n(x)$ by $q_{n-2}(x)$, we obtain ($n \geq 2$)

$$2n\int_{-1}^{1} x^2(1-x^2) q_{n-2}^2(x)\,dx \geq \int_{-1}^{1} (1-x^2)^2 q_{n-2}^2(x)\,dx - \frac{1}{n}\int_{-1}^{1} (1-x^2)^2 \sum_{k=1}^{n-2} \frac{(1-x_k^2)}{(x-x_k)^2} q_{n-2}^2(x)\,dx. \tag{2.20}$$

IDENTITY 2.5. Let $P_n(x)$ be any polynomial of degree n; then we have

$$(2n+5)(2n+1)\int_{-1}^{1} x^4 P_n^2(x)\,dx + 6\int_{-1}^{1} x^2 P_n^2(x)\,dx - 3\int_{-1}^{1} P_n^2(x)\,dx$$
$$= \frac{2(2n+1)}{(n+2)} I_{2,n} + \frac{2(5n+7)}{(n+1)(n+2)} I_{0,n}$$
$$- \frac{(5n+7)}{(n+1)(n+2)} \int_{-1}^{1} (1-x^2)^2 P_n^2(x) \sum_{k=1}^{n} \frac{(1-x_k^2)}{(x-x_k)^2}\,dx$$
$$- \frac{2(n+2)}{(n+1)} \int_{-1}^{1} x^2(1-x^2) \sum_{k=1}^{n} \frac{1-x_k^2}{(x-x_k)^2} P_n^2(x)\,dx,$$

where

$$I_{2r,n} = \int_{-1}^{1} x^{2r}\left(nxP_n(x) + (1-x^2)P_n'(x)\right)^2 dx.$$

For the proof of this identity one uses (2.3) for $r = 1$ and $r = 2$. Replacing n by $n-2$ and $P_n(x)$ by $q_{n-2}(x)$, we obtain ($n > 2$)

$$4n(n-1)\int_{-1}^{1} x^4 q_{n-2}^2(x)\,dx > 3\int_{-1}^{1} (1-x^2)^2 q_{n-2}^2(x)\,dx$$
$$\text{(2.21)}\qquad - \frac{(5n-3)}{n(n-1)} \int_{-1}^{1} (1-x^2)^2 \sum_{k=1}^{n-2} \frac{(1-x_k^2)}{(x-x_k)^2} q_{n-2}^2(x)\,dx$$
$$- \frac{2n}{(n-1)} \int_{-1}^{1} x^2(1-x^2) \sum_{k=1}^{n-2} \frac{(1-x_k^2)}{(x-x_k)^2} q_{n-2}^2(x)\,dx.$$

IDENTITY 2.6. Let $P_n(x)$ be any polynomial of degree n; then we have

$$2\int_{-1}^{1} x^2 P_n'^2(x)\,dx = \int_{-1}^{1} P_n^2(x)\,dx$$
$$\text{(2.22)}\qquad + \int_{-1}^{1} x^2\left(P_n'^2(x) - P_n(x)P_n''(x)\right) dx$$
$$+ \left(P_n(1)P_n'(1) - P_n^2(1)\right) - \left(P_n(-1)P_n'(-1) + P_n^2(-1)\right).$$

PROOF. On integrating by parts, we obtain

$$\int_{-1}^{1} P_n^2(x)\,dx = P_n^2(1) + P_n^2(-1) - \int_{-1}^{1} x \frac{d}{dx} P_n^2(x)\,dx$$
$$= P_n^2(1) + P_n^2(-1) - \left(P_n(1)P_n'(1) - P_n(-1)P_n'(-1)\right)$$
$$+ \int_{-1}^{1} x^2\left(P_n'^2(x) + P_n(x)P_n''(x)\right) dx.$$

But this is equivalent to (2.22).

LEMMA 2.1. *Let* $P_n \in H_n$, $P_n(1) = P_n(-1) = 0$, *and*

$$P_n(x) = (1-x^2)q_{n-2}(x),$$

where $q_{n-2}(x)$ *is a polynomial of degree* $n-2$; *then*

$$\int_{-1}^{1} x^2\left(P_n'^2(x) - P_n(x)P_n''(x)\right) dx$$

(2.23)
$$\geqslant \left(\frac{1}{2} + \frac{3}{2(n-1)}\right)\int_{-1}^{1} P_n^2(x)\, dx - \frac{(n+1)}{2(n-1)^2}\int_{-1}^{1} P_n^2(x)\sum_{k=1}^{n-2}\frac{(1-x_k^2)}{(x-x_k)^2}\, dx,$$

equality for $P_n(x) = (1-x^2)^m$, $n = 2m$.

PROOF. On using (2.4)–(2.6) we obtain

$$\int_{-1}^{1} x^2\left(P_n'^2(x) - P_n(x)P_n''(x)\right) dx = \int_{-1}^{1} x^2P_n^2(x)\left(\sum_{k=1}^{n}\frac{1}{(x-x_k)^2}\right) dx$$
$$= n\int_{-1}^{1}\frac{x^2P_n^2(x)\, dx}{(1-x^2)} - 2\int_{-1}^{1}\frac{P_n(x)P_n'(x)x^3}{(1-x^2)}\, dx$$
$$+ \int_{-1}^{1} x^2(1-x^2)q_{n-2}^2(x)\sum_{k=1}^{n-2}\frac{(1-x_k^2)}{(x-x_k)^2}\, dx.$$

But

$$-\int_{-1}^{1}\frac{x^3}{(1-x^2)}\frac{d}{dx}P_n^2(x)\, dx = 3\int_{-1}^{1}\frac{x^2}{(1-x^2)}P_n^2(x)\, dx + 2\int_{-1}^{1}\frac{x^4}{(1-x^2)^2}P_n^2(x)\, dx.$$

Hence on using (2.20) and (2.21) we obtain

$$\int_{-1}^{1} x^2\left(P_n'^2(x) - P_n(x)P_n''(x)\right) dx$$
$$= (n+3)\int_{-1}^{1} x^2(1-x^2)q_{n-2}^2(x)\, dx$$
$$+ 2\int_{-1}^{1} x^4q_{n-2}^2(x)\, dx$$
$$+ \int_{-1}^{1} x^2(1-x^2)q_{n-2}^2(x)\sum_{k=1}^{n-2}\frac{1-x_k^2}{(x-x_k)^2}\, dx$$
$$\geqslant \left(\frac{n+3}{2n} + \frac{3}{2n(n-1)}\right)\int_{-1}^{1} P_n^2(x)\, dx$$
$$- \frac{(n+1)}{2(n-1)^2}\int_{-1}^{1} P_n^2(x)\sum_{k=1}^{n-2}\frac{(1-x_k^2)}{(x-x_k)^2}\, dx.$$

3. Proof of Theorem 1. From (1.1) we have

$$
\begin{aligned}
2\int_{-1}^{1}(1-x^2)P_n'^2(x)\,dx &= n\int_{-1}^{1}P_n^2(x)\,dx \\
&\quad + \int_{-1}^{1}P_n^2(x)\sum_{k=1}^{n-2}\frac{(1-x_k^2)}{(x-x_k)^2}\,dx
\end{aligned}
\tag{3.1}
$$

(we used here $P_n(1) = P_n(-1) = 0$). On using (2.22) and (2.23) we obtain

$$
\begin{aligned}
2\int_{-1}^{1}x^2P_n'^2(x)\,dx &\geqslant \int_{-1}^{1}P_n^2(x)\,dx + \left(\frac{1}{2}+\frac{3}{2(n-1)}\right)\int_{-1}^{1}P_n^2(x)\,dx \\
&\quad - \frac{(n+1)}{2(n-1)^2}\int_{-1}^{1}P_n^2(x)\sum_{k=1}^{n-2}\frac{(1-x_k^2)}{(x-x_k)^2}\,dx.
\end{aligned}
\tag{3.2}
$$

On adding (3.1) and (3.2) we obtain ($n \geqslant 2$)

$$
\begin{aligned}
2\int_{-1}^{1}P_n'^2(x)\,dx &\geqslant \left(n+\frac{3}{2}+\frac{3}{2(n-1)}\right)\int_{-1}^{1}P_n^2(x)\,dx \\
&\quad + \left(1-\frac{n+1}{2(n-1)^2}\right)\int_{-1}^{1}P_n^2(x)\sum_{k=1}^{n}\frac{(1-x_k^2)}{(x-x_k)^2}\,dx \\
&\geqslant \left(n+\frac{3}{2}-\frac{3}{2(n-1)}\right)\int_{-1}^{1}P_n^2(x)\,dx.
\end{aligned}
$$

This proves Theorem 1.

PROOF OF THEOREM 2. Proof of (1.5)–(1.7) is similar. So we give the details for (1.5) only. From (2.1) and (2.13) we obtain

$$
\begin{aligned}
2\int_{-1}^{1}(1-x^4)P_n'^2(x)\,dx &= (n+1)\int_{-1}^{1}P_n^2(x)\,dx \\
&\quad + (n-3)\int_{-1}^{1}x^2P_n^2(x)\,dx + S_0 + S_2 \\
&\geqslant \left(n+1+\frac{n-3}{2n+3}\right)\int_{-1}^{1}P_n^2(x)\,dx \\
&\quad + \frac{(n-3)(S_2-S_0)}{(n+1)(2n+3)} + S_0 + S_2 \\
&\geqslant \left(n+\frac{3n}{2n+3}\right)\int_{-1}^{1}P_n^2(x)\,dx + \frac{2(n^2+2n+3)}{(n+1)(2n+3)}S_0 \\
&\quad + \frac{n(n+6)}{(n+1)(2n+3)}S_2 \\
&\geqslant \left(n+\frac{3n}{2n+3}\right)\int_{-1}^{1}P_n^2(x)\,dx.
\end{aligned}
$$

This proves Theorem 2.

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