A COMMUTANT OF AN UNBOUNDED OPERATOR ALGEBRA

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Abstract. A commutant $\mathfrak{A}'$ and bicommutant $\mathfrak{A}^{\text{cc}}$ of an unbounded operator algebra $\mathfrak{A}$ called a $\#$-algebra are defined. The first purpose of this paper is to investigate whether the bicommutant $\mathfrak{A}^{\text{cc}}$ of a $\#$-algebra $\mathfrak{A}$ is an $EW^*$-algebra, as defined in [6], or not. The second purpose is to investigate the relation between $\mathfrak{A}^{\text{cc}}$ and topologies on a $\#$-algebra $\mathfrak{A}$.

In this paper let $\mathcal{H}$ be a pre-Hilbert space with an inner product $(\cdot | \cdot)$ and $\mathfrak{A}$ the completion of $\mathcal{H}$. Let $\mathcal{L}(\mathcal{H})$ denote the set of all linear operators on $\mathcal{H}$ and $\mathcal{L}^* (\mathcal{H})$ the set $\{A \in \mathcal{L}(\mathcal{H}); A^* \mathcal{H} \subset \mathcal{H}\}$. Every element $A$ of $\mathcal{L}^* (\mathcal{H})$ is a closable operator on $\mathfrak{A}$ with the domain $\mathcal{H}$. For each $A \in \mathcal{L}^* (\mathcal{H})$, putting $A^\# = A^* / \mathcal{H}$ (the restriction of $A^*$ onto $\mathcal{H}$), the map $A \rightarrow A^\#$ is an involution on $\mathcal{L}^* (\mathcal{H})$ and $\mathcal{L}^* (\mathcal{H})$ is an algebra of operators on $\mathcal{H}$ with the involution $\#$.

If $\mathfrak{A}$ is a $\#$-subalgebra of $\mathcal{L}^* (\mathcal{H})$, then it is called a $\#$-algebra on $\mathcal{H}$. In particular, $\mathcal{L}^* (\mathcal{H})$ is called a maximal $\#$-algebra on $\mathcal{H}$. We denote by $\overline{S}$ the smallest closed extension of $S \in \mathfrak{A}$ and by $\overline{\mathfrak{A}}$ the set $\{S; S \in \mathfrak{A}\}$. We set

$$\mathfrak{A}^b = \{A \in \mathfrak{A}; \overline{A} \in \mathcal{B}(\mathfrak{A})\},$$

where $\mathcal{B}(\mathfrak{A})$ denotes the set of all bounded linear operators on $\mathfrak{A}$, and call it the bounded part of $\mathfrak{A}$. A $\#$-algebra $\mathfrak{A}$ is called pure if $\mathfrak{A} \neq \mathfrak{A}^b$. A $\#$-algebra $\mathfrak{A}$ on $\mathcal{H}$ is said to be symmetric if $\mathfrak{A}$ has an identity operator $I$ and $(I + A^\# A)^{-1} \in \mathfrak{A}$ for all $A \in \mathfrak{A}$. A symmetric $\#$-algebra $\mathfrak{A}$ on $\mathcal{H}$ is called an $EW^*$-algebra on $\mathcal{H}$ over $\mathfrak{A}^b$ if $\mathfrak{A}^b$ is a von Neumann algebra.

A $\#$-algebra $\mathfrak{A}$ on $\mathcal{H}$ is said to be closed (resp. selfadjoint) if

$$\mathcal{H} = \bigcap_{A \in \mathfrak{A}} \mathcal{D}(A) \quad (\text{resp. } \mathcal{H} = \bigcap_{A \in \mathfrak{A}} \mathcal{D}(A^*)),$$

It is easy to show that if $\mathfrak{A}$ is a selfadjoint $\#$-algebra on $\mathcal{H}$ then it is closed. By [6, Proposition 2.6], if $\mathfrak{A}$ is a closed symmetric $\#$-algebra, then it is selfadjoint.

In [6] we defined the commutant $\mathfrak{A}'$ of a $\#$-algebra $\mathfrak{A}$ on $\mathcal{H}$ as follows:

$$\mathfrak{A}' = \{C \in \mathcal{B}(\mathfrak{A}); (CA|A^\# \eta) = (C \xi|A^\# \eta) \quad \text{for all } A \in \mathfrak{A} \text{ and } \xi, \eta \in \mathcal{H}\}.$$

From [6, Proposition 2.8] if $\mathfrak{A}$ is a selfadjoint $\#$-algebra on $\mathcal{H}$ then $\mathfrak{A}'$ is a von
Neumann algebra. Furthermore, for each \( C \in \mathcal{A} \), \( A \in \mathcal{A} \) and \( \xi \in \mathcal{D} \) we have \( C \mathcal{D} \subset \mathcal{D} \) and \( CA\xi = AC\xi \). We define a new commutant \( \mathcal{A}' \) and bicommutant \( \mathcal{A}^{bc} \) as follows:

\[
\mathcal{A}' = \left\{ S \in L^\infty(\mathcal{D}); SA = AS \text{ for all } A \in \mathcal{A} \right\},
\]

\[
\mathcal{A}^{bc} = \left\{ A \in L^\infty(\mathcal{D}); SA = AS \text{ for all } S \in \mathcal{A}' \right\}.
\]

It is immediately shown that \( \mathcal{A}', \mathcal{A}^{bc} \) are \( * \)-algebras on \( \mathcal{D} \) and \( \mathcal{A}^{bc} \subset \mathcal{A} \).

**Lemma 1.** If \( \mathcal{A} \) is a selfadjoint \( * \)-algebra on \( \mathcal{D} \), then:

1. \( \mathcal{T} \) is affiliated with \( (\mathcal{A}_b)' \) (written \( T\eta(\mathcal{A}_b)' \)) for each \( T \in \mathcal{A}' \) and \( (\mathcal{A}_b)_b = \mathcal{A}' \subset (\mathcal{A}_b)' \);
2. \( A\eta\mathcal{A}'' \) for each \( A \in \mathcal{A}^{bc} \) and \( \mathcal{A}_b \subset (\mathcal{A}^{bc})_b \subset \mathcal{A}'' \).

**Lemma 2.** If \( \mathcal{A} \) is a symmetric \( * \)-algebra on \( \mathcal{D} \), then \( (\mathcal{A}_b)' = \mathcal{A}' \) and \( (\mathcal{A}_b)_b = \mathcal{A}'' \).

**Proof.** Clearly, \( \mathcal{A}' \subset (\mathcal{A}_b)' \). Suppose that \( C \in (\mathcal{A}_b)' \). Let \( A \in \mathcal{A}_b := \{ A \in \mathcal{A}; A^* = A \} \). Since \( \mathcal{A} \) is symmetric, it is easily shown that \( (I + A^2)^{-1}, A(I + A^2)^{-1} \in \mathcal{A}_b \). For each \( \xi, \eta \in \mathcal{D} \) we have

\[
\left( CA(I + A^2)^{-1}\xi|\eta\right) = \left( A(I + A^2)^{-1}C\xi|\eta\right) = \left( AC(I + A^2)^{-1}\xi|\eta\right).
\]

Since \( (I + A^2)^{-1}\mathcal{D} = \mathcal{D} \), we get \( C \in \mathcal{A}' \). Thus, \( (\mathcal{A}_b)' = \mathcal{A}' \).

**Proposition 1.** Let \( \mathcal{A} \) be a closed symmetric \( * \)-algebra on \( \mathcal{D} \). Then:

1. \( \mathcal{A}' \) is an \( EW^* \)-algebra on \( \mathcal{D} \) over \( \mathcal{A}' \);
2. \( \mathcal{A}^{bc} = \{ A \in L^\infty(\mathcal{D}); A\eta\mathcal{A}'' = (\mathcal{A}_b)_b \} \);
3. if \( (\mathcal{A}_b)' \subset \mathcal{D} \), then \( \mathcal{A}^{bc} \) is a closed \( EW^* \)-algebra on \( \mathcal{D} \) over \( (\mathcal{A}_b)_b \).

**Proof.** From [6, Proposition 2.6] \( \mathcal{A} \) is a selfadjoint \( * \)-algebra on \( \mathcal{D} \). Hence this follows from Lemma 1.2.

**Corollary.** If \( \mathcal{A} \) is a closed \( EW^* \)-algebra on \( \mathcal{D} \), then \( \mathcal{A}' \) is an \( EW^* \)-algebra on \( \mathcal{D} \) over \( \mathcal{A}' \) and \( \mathcal{A}^{bc} \) is a closed \( EW^* \)-algebra on \( \mathcal{D} \) over \( (\mathcal{A}_b)_b \).

In this paper let \( \mathcal{D} \) be an unbounded Hilbert algebra over \( \mathcal{D}_0 \) and \( b(\mathcal{D}_0) \) the completion of \( \mathcal{D}_0 \). For the basic definitions and facts of unbounded Hilbert algebras the reader is referred to [7], [8]. Let \( \mathcal{H}_0(\mathcal{D}_0) \) (resp. \( \mathcal{V}_0(\mathcal{D}_0) \)) be the left (resp. right) von Neumann algebra of the Hilbert algebra \( \mathcal{D}_0 \). Let \( \pi_0 \) (resp. \( \pi_0^* \)) be the left (resp. right) regular representation of \( \mathcal{D}_0 \) and \( \phi_0 \) (resp. \( \phi_0^* \)) the natural trace on \( \mathcal{H}_0(\mathcal{D}_0)^* \) (resp. \( \mathcal{V}_0(\mathcal{D}_0)^* \)). For each \( x \in b(\mathcal{D}_0) \) we define

\[
\pi_0(x)\xi = \overline{\pi_0(\xi)}^* x, \quad \pi_0^*(x)\xi = \pi_0(\overline{\xi}) x \quad (\xi \in \mathcal{D}_0).
\]

Then \( \pi_0(x) \) and \( \pi_0^*(x) \) are linear operators on \( b(\mathcal{D}_0) \) with domain \( \mathcal{D}_0 \) and \( \pi_0(\overline{x}) = \pi_0(x)^* \), \( \pi_0^*(\overline{x}) = \pi_0(x)^* \). Putting \( (\mathcal{D}_0)_b = \{ x \in b(\mathcal{D}_0); \pi_0(x) \in b(\mathcal{H}(\mathcal{D}_0)) \} \), \( (\mathcal{D}_0)_b \) is a Hilbert algebra containing \( \mathcal{D}_0 \) and is called the maximal Hilbert algebra of \( \mathcal{D}_0 \) in \( b(\mathcal{D}_0) \).

Let \( \mathcal{M} \) (resp. \( \mathcal{M}^+ \)) be the set of all measurable (resp. positive measurable)
operators on \( b(\mathcal{D}_0) \) with respect to \( \mathcal{D}_0(\mathcal{D}_0) \). For each \( T \in \mathfrak{M}^+ \) we set

\[
\mu_0(T) = \sup \left\{ \phi_0(\tau_0(\xi)); 0 < \tau_0(\xi) < T, \xi \in (\mathcal{D}_0)_b \right\},
\]

\[
L^p(\phi_0) = \left\{ T \in \mathfrak{M}; \|T\|_p = \mu_0((|T|^p)^{1/p}) < \infty \right\}, \quad 1 \leq p < \infty,
\]

\[
L^\infty(\phi_0) = \mathcal{D}_0(\mathcal{D}_0).
\]

We define \( L^p_+ \)-spaces with respect to \( \phi_0 \) and \( \mathcal{D}_0 \) as follows:

\[
L^p_*(\mathcal{D}_0) = \bigcap_{2 \leq p < \infty} L^p(\phi_0), \quad L^\infty_*(\mathcal{D}_0) = \left\{ x \in \mathfrak{H}; \tau_0(x) \in L^\infty_*(\phi_0) \right\},
\]

respectively. By [7, Theorem 3.9] \( L^\infty_*(\mathcal{D}_0) \) is maximal among unbounded Hilbert algebras containing \( \mathcal{D}_0 \) and is called the maximal unbounded Hilbert algebra of \( \mathcal{D}_0 \). Let \( \pi^\#_2 \) (resp. \( (\pi^\#_2)' \)) be the left (resp. right) regular representation (i.e., \( \pi^\#_2(x)y = xy, (\pi^\#_2)'(x)y = yx, x, y \in L^\infty_*(\mathcal{D}_0) \)) of \( L^\infty_*(\mathcal{D}_0) \). We set

\[
\mathcal{U}_0(\mathcal{D}_0)/L^\infty_*(\mathcal{D}_0) = \left\{ T/L^\infty_*(\mathcal{D}_0); T \in \mathcal{U}_0(\mathcal{D}_0) \right\},
\]

\[
\mathcal{V}_0(\mathcal{D}_0)/L^\infty_*(\mathcal{D}_0) = \left\{ T'/L^\infty_*(\mathcal{D}_0); T' \in \mathcal{V}_0(\mathcal{D}_0) \right\}.
\]

Then \( \pi^\#_2(\mathcal{D}_0), (\pi^\#_2)'(\mathcal{D}_0), \mathcal{U}_0(\mathcal{D}_0)/L^\infty_*(\mathcal{D}_0) \) and \( \mathcal{V}_0(\mathcal{D}_0)/L^\infty_*(\mathcal{D}_0) \) are \#-algebras on \( L^\infty_*(\mathcal{D}_0) \). We denote by \( \mathcal{U}(\mathcal{D}_0) \) (resp. \( \mathcal{V}(\mathcal{D}_0) \)) a \#-algebra on \( L^\infty_*(\mathcal{D}_0) \) generated by \( \pi^\#_2(\mathcal{D}_0) \) and \( \mathcal{U}_0(\mathcal{D}_0)/L^\infty_*(\mathcal{D}_0) \) (resp. \( (\pi^\#_2)'(\mathcal{D}_0) \) and \( \mathcal{V}_0(\mathcal{D}_0)/L^\infty_*(\mathcal{D}_0) \)). Then \( \mathcal{U}(\mathcal{D}_0) \) (resp. \( \mathcal{V}(\mathcal{D}_0) \)) is an \( EW^\# \)-algebra on \( L^\infty_*(\mathcal{D}_0) \) over \( \mathcal{U}_0(\mathcal{D}_0) \) (resp. \( \mathcal{V}_0(\mathcal{D}_0) \)) and is called the left (resp. right) \( EW^\# \)-algebra of \( \mathcal{D}_0 \).

**Theorem 1.** Suppose that \( \mathfrak{H}(\mathcal{D}_0) \) is not a Hilbert algebra, i.e., \( \mathfrak{H}(\mathcal{D}_0) \neq (\mathcal{D}_0)_b \).

Then:

1. \( \pi^\#_2(\mathcal{D}_0)' \) is a pure \( EW^\# \)-algebra on \( L^\infty_*(\mathcal{D}_0) \) over \( \mathcal{V}_0(\mathcal{D}_0) \) such that
   \[
   \pi^\#_2(\mathcal{D}_0)' = \left\{ T \in L^\#_*(L^\infty_*(\mathcal{D}_0)); \eta \mathcal{V}_0(\mathcal{D}_0) \supset \mathcal{V}(L^\infty_*(\mathcal{D}_0)) \right\};
   \]

2. \( \pi^\#_2(\mathcal{D}_0)' \) is a pure \( EW^\# \)-algebra on \( L^\infty_*(\mathcal{D}_0) \) over \( \mathcal{U}_0(\mathcal{D}_0) \) such that
   \[
   \pi^\#_2(\mathcal{D}_0)' = \left\{ A \in L^\#_*(L^\infty_*(\mathcal{D}_0)); \eta \mathcal{U}_0(\mathcal{D}_0) \supset \mathcal{U}(L^\infty_*(\mathcal{D}_0)) \right\}.
   \]

**Proof.** (1) It is easily proved that \( \pi^\#_2(\mathcal{D}_0)' = \left\{ T \in L^\#_*(L^\infty_*(\mathcal{D}_0)); \eta \mathcal{V}_0(\mathcal{D}_0) \right\} \). Since \( \mathcal{V}_0(\mathcal{D}_0)L^\infty_*(\mathcal{D}_0) \subset L^\infty_*(\mathcal{D}_0) \) and \( (\pi^\#_2)'(L^\infty_*(\mathcal{D}_0)L^\infty_*(\mathcal{D}_0)) \subset L^\infty_*(\mathcal{D}_0) \) and \( (\pi^\#_2)'(L^\infty_*(\mathcal{D}_0)) \) are contained in \( \pi^\#_2(\mathcal{D}_0)' \). Hence \( \mathcal{V}(L^\infty_*(\mathcal{D}_0)) \subset \pi^\#_2(\mathcal{D}_0)' \). Furthermore, \( \mathcal{V}_0(\mathcal{D}_0) \supset \mathcal{V}_0(\mathcal{D}_0) \). Thus \( \pi^\#_2(\mathcal{D}_0)' \) is an \( EW^\# \)-algebra on \( L^\infty_*(\mathcal{D}_0) \) over \( \mathcal{V}_0(\mathcal{D}_0) \) containing \( \mathcal{V}(L^\infty_*(\mathcal{D}_0)) \). By [8, Theorem 3.4], \( \mathcal{V}(L^\infty_*(\mathcal{D}_0)) \) is pure and it follows that \( \pi^\#_2(\mathcal{D}_0)' \) is pure.

(2) This is proved in the same way as (1).

**Corollary.** If \( \mathcal{D}_0 \) is pure, then
\[ \pi_2^\omega(\mathcal{D}_0)^c = \pi_2^\omega(\mathcal{D})^c = \pi_2^\omega(L_2^\omega(\mathcal{D}_0))^c \]

and

\[ \pi_2^\omega(\mathcal{D}_0)^c = \pi_2^\omega(\mathcal{D})^c = \pi_2^\omega(L_2^\omega(\mathcal{D}_0))^c. \]

Furthermore, \( \pi_2^\omega(\mathcal{D}_0)^c \) (resp. \( \pi_2^\omega(\mathcal{D}_0)^c \)) is maximal among \( \text{EW}^\omega \)-algebras on \( L_2^\omega(\mathcal{D}_0) \) over \( \mathcal{M}_0(\mathcal{D}_0) \) (resp. \( \mathcal{M}_0(\mathcal{D}_0) \)).

We call \( \pi_2^\omega(\mathcal{D}_0)^c \) (resp. \( \pi_2^\omega(\mathcal{D}_0)^c \)) the maximal left (resp. right) \( \text{EW}^\omega \)-algebra of \( \mathcal{D}_0 \) and denote it by \( \mathcal{M}_l(\mathcal{D}_0) \) (resp. \( \mathcal{M}_r(\mathcal{D}_0) \)).

**Proposition 2.** Let \( \mathcal{M}(\phi_0) \) (resp. \( \mathcal{M}(\phi_0) \)) be the set of all \( \phi_0 \)-measurable (resp. \( \phi_0 \)-measurable) operators. Then:

1. \( \mathcal{U}(L_2^\omega(\mathcal{D}_0)) = \{ A \in L_2^\omega(\mathcal{D}_0); \tilde{A} \in \mathcal{M}(\phi_0) \}; \)
2. \( \mathcal{V}(L_2^\omega(\mathcal{D}_0)) = \{ T \in L_2^\omega(\mathcal{D}_0); \tilde{T} \in \mathcal{M}(\phi_0) \}. \)

**Proof.** (1) Suppose that \( A \in L_2^\omega(\mathcal{D}_0) \) and \( A \in \mathcal{M}(\phi_0) \). Let \( \tilde{A} = U|A| \) be the polar decomposition of \( \tilde{A} \) and \( |A| = \int_0^\infty \lambda \, dE(\lambda) \) the spectral resolution of \( |A| \). Since \( A \) is \( \phi_0 \)-measurable, \( |A| \) is \( \phi_0 \)-measurable, and it follows that \( E(\lambda_0):= I - E(\lambda_0) \in \pi_2^\omega(\mathcal{D}_0) \) for some \( \lambda_0 > 0 \), i.e., there exists an element \( e_{\lambda_0} \in \mathcal{D}_0 \) such that \( E(\lambda_0) = \pi_2^\omega(e_{\lambda_0}) \). Putting

\[ |A| = \frac{1}{\lambda_0}L_2^\omega(\mathcal{D}_0), \]

\[ A_0 = \int_0^{\lambda_0} \lambda \, dE(\lambda)/L_2^\omega(\mathcal{D}_0) \) and \( U_0 = U/L_2^\omega(\mathcal{D}_0), \)

\[ |A| = L_2^\omega(\mathcal{D}_0) \) and \( A_0, U_0 \in \mathcal{M}(\phi_0)/L_2^\omega(\mathcal{D}_0). \)

Furthermore,

\[ A = U_0A_0 + \pi_2^\omega(U_0|A|e_{\lambda_0}) \in \mathcal{M}(\phi_0)/L_2^\omega(\mathcal{D}_0) + \pi_2^\omega(L_2^\omega(\mathcal{D}_0)). \]

Hence, \( A \in \mathcal{U}(L_2^\omega(\mathcal{D}_0)). \) Thus, \( \{ A \in L_2^\omega(\mathcal{D}_0); \tilde{A} \in \mathcal{M}(\phi_0) \} \subset \mathcal{U}(L_2^\omega(\mathcal{D}_0)). \) The reverse inclusion is obvious.

(2) This is proved in the same way as (1).

**Theorem 2.** If \( \mathcal{D}_0 \) has an identity or \( b(\mathcal{D}_0) \) is separable, then \( \mathcal{M}_l(\mathcal{D}_0) = \mathcal{U}(L_2^\omega(\mathcal{D}_0)) \) and \( \mathcal{M}_r(\mathcal{D}_0) = \mathcal{V}(L_2^\omega(\mathcal{D}_0)). \)

**Proof.** If \( \mathcal{D}_0 \) has an identity, then this is easily proved. Suppose that \( b(\mathcal{D}_0) \) is separable. P. G. Dixon [4, Theorem 5.3] has proved that each \( \text{EW}^\omega \)-algebra \( \mathfrak{A} \) over a von Neumann algebra \( \mathfrak{B} \) is contained in the algebra \( \mathcal{M}(\mathfrak{A}) \) of all measurable operators with respect to \( \mathfrak{A} \). From Theorem 1, \( \mathcal{M}_l(\mathcal{D}_0) \) is an \( \text{EW}^\omega \)-algebra over \( \mathcal{D}_0(\mathcal{D}_0) \). Hence, \( \mathcal{M}_l(\mathcal{D}_0) \subset \mathcal{M}(\mathcal{D}_0(\mathcal{D}_0)). \) Suppose that \( A \in \mathcal{M}_l(\mathcal{D}_0) \). Let \( \tilde{A} = U|A| \) be the polar decomposition of \( \tilde{A} \) and \( |A| = \int_0^{\lambda_0} \lambda \, dE(\lambda) \) the spectral resolution of \( |A| \). Since \( A \) is measurable, \( E_0 := E(\lambda_0) \) is a finite projection for some \( \lambda_0 > 0 \) and it follows that \( \mathcal{D}_0(\mathcal{D}_0)E_0 \) is a finite von Neumann algebra. Furthermore, since \( b(\mathcal{D}_0) \) is separable, \( \mathcal{D}_0(\mathcal{D}_0)E_0 \) is \( \sigma \)-finite. From [3, §6, Proposition 9], there exists a faithful normal finite trace \( \chi_0 \) on \( \mathcal{D}_0(\mathcal{D}_0)E_0 \) and it follows that there exists an isomorphism \( \Psi \) of \( \mathcal{D}_0(\mathcal{D}_0)E_0 \) onto a standard von Neumann algebra \( \mathfrak{A}_0 \) such that \( \chi_0(T) = \chi_0(\tilde{T}) \).
\(x(\Psi(T))\) for every \(T \in \mathcal{U}_0(\mathfrak{D}_0)_E^+\), where \(x\) denotes the natural trace on \(\mathfrak{A}_0^+\) \cite[§6, Theorem 2]{3}. We set

\[(\phi_0)_E^+(T) = \phi_0(TE_0), \quad T \in \mathcal{U}_0(\mathfrak{D}_0)_E^+.\]

Then it is easily proved that \((\phi_0)_E^+\) is a faithful normal semifinite trace on \(\mathcal{U}_0(\mathfrak{D}_0)_E^+\). Hence, from \cite[§6, Theorem 2]{3} there exists an isomorphism \(\Phi\) of \(\mathcal{U}_0(\mathfrak{D}_0)_E^+\) onto a standard von Neumann algebra \(\mathfrak{B}_0\) such that \((\phi_0)_E^+(T) = \Phi(\Phi(T))\) for every \(T \in \mathcal{U}_0(\mathfrak{D}_0)_E^+\), where \(\Phi\) denotes the natural trace on \(\mathfrak{B}_0^+\).

Then the standard von Neumann algebras \(\mathfrak{A}_0\) and \(\mathfrak{B}_0\) are isomorphic. From \cite[§6, Theorem 4]{3} \(\mathfrak{A}_0\) and \(\mathfrak{B}_0\) are spatially isomorphic. Since \(x(T) < \infty\) for all \(T \in \mathfrak{A}_0^+\), \(\phi(T) < \infty\) for all \(T \in \mathfrak{B}_0^+\). Hence, \(\phi_0(E(\lambda_0)^{-1}) = \phi(\Phi(I)) < \infty\).

Thus we can show that if \(S\) is measurable with respect to \(\mathcal{U}_0(\mathfrak{D}_0)_E^+\) then \(S\) is \(\phi_0^\text{-measurable}\). Hence, Theorem 2 follows from Proposition 2.

Next we shall investigate the relation between the commutants \(\mathfrak{A}^c\), \(\mathfrak{A}^c\) of a \#-algebra \(\mathfrak{A}\) on \(\mathfrak{D}\) and topologies on \(\mathfrak{A}\). The locally convex topology induced by seminorms: \(P_{\xi, n}(T) = |(T\xi|\eta)|\) \((\xi, \eta \in \mathfrak{D})\) is called the weak topology on \(\mathfrak{A}\). Let \(\mathfrak{B}\) be a \#-algebra on \(\mathfrak{D}\) containing \(\mathfrak{A}\). We set

\[\mathcal{D}_\infty(\mathfrak{B}) = \left\{\xi_\infty = (\xi_1, \xi_2, \ldots, \xi_n, \ldots); \xi_n \in \mathfrak{D} (n = 1, 2, \ldots)\right\}\]

and \(\sum_{n=1}^\infty \|T\xi_n\|^2 < \infty\) for all \(T \in \mathfrak{B}\).\]

For each \(\xi_\infty = (\xi_1, \xi_2, \ldots, \xi_n, \ldots)\) and \(\eta_\infty = (\eta_1, \eta_2, \ldots, \eta_n, \ldots)\) in \(\mathcal{D}_\infty(\mathfrak{B})\), \(P_{\xi, n}(T) = |(T\xi|\eta)|\) \((T \in \mathfrak{A})\) is a seminorm on \(\mathfrak{A}\). The locally convex topology on \(\mathfrak{A}\) induced by the family \(\{P_{\xi, n}(\cdot); \xi_\infty, \eta_\infty \in \mathcal{D}_\infty(\mathfrak{B})\}\) of the seminorms is called the \((\mathfrak{B})\)-\(\sigma\)-weak topology on \(\mathfrak{A}\). In particular, the \((\mathcal{L}^\#(\mathfrak{D}))\)-\(\sigma\)-weak topology on \(\mathfrak{A}\) is simply called the \(\sigma\)-weak topology on \(\mathfrak{A}\). It is easy to check that \(\mathfrak{A}\) is a locally convex \#-algebra under the involution \# and weak topology (or, \((\mathfrak{B})\)-\(\sigma\)-weak topology). The strong topology is the locally convex topology induced by seminorms: \(P_{\xi}(T) = \|T\xi\|\) \((\xi \in \mathfrak{D})\). For each \(\xi_\infty = (\xi_1, \xi_2, \ldots, \xi_n, \ldots) \in \mathcal{D}_\infty(\mathfrak{B})\), \(P_{\xi, n}(T) := (\sum_{n=1}^\infty \|T\xi_n\|^2)^{1/2}\) \((T \in \mathfrak{A})\) is a seminorm on \(\mathfrak{A}\). The locally convex topology induced by the seminorms \(\{P_{\xi, n}(\cdot); \xi_\infty \in \mathcal{D}_\infty(\mathfrak{B})\}\) is called the \((\mathfrak{B})\)-\(\sigma\)-strong topology on \(\mathfrak{A}\). In particular, the \((\mathcal{L}^\#(\mathfrak{D}))\)-\(\sigma\)-strong topology on \(\mathfrak{A}\) is simply called the \(\sigma\)-strong topology on \(\mathfrak{A}\).

**Proposition 3.** If \(\mathfrak{A}\) is a \#-algebra on \(\mathfrak{D}\), then \(\mathfrak{A}^c\) and \(\mathfrak{A}^c\) are closed in \(\mathcal{L}^\#(\mathfrak{D})\) under the weak topology.

**Corollary.** \(\mathfrak{A}^c(\mathfrak{D}_0)\) and \(\mathfrak{A}^c(\mathfrak{D}_0)\) are closed in \(\mathcal{L}^\#(L^2(\mathfrak{D}_0))\) under the weak topology.

**Theorem 3.** If \(\mathfrak{A}\) is a closed symmetric \#-algebra on \(\mathfrak{D}\), then the following algebras, (1) \(\cdots\) (6), coincide with \(\mathfrak{A}^c\):

1. the weak closure \([\mathfrak{A}_b]^*\) of \(\mathfrak{A}_b\) in \(\mathcal{L}^\#(\mathfrak{D})\);
2. the strong closure \([\mathfrak{A}_b]^*\) of \(\mathfrak{A}_b\) in \(\mathcal{L}^\#(\mathfrak{D})\);
(3) the $\sigma$-weak closure $[\mathcal{B}_b]^\omega$ of $\mathcal{B}_b$ in $L^\#(\mathfrak{D})$;
(4) the $\sigma$-strong closure $[\mathcal{B}_b]^\sigma$ of $\mathcal{B}_b$ in $L^\#(\mathfrak{D})$;
(5) the $(\mathfrak{R}^c)$-$\sigma$-weak closure $[\mathcal{B}_b]^\omega_{\mathfrak{R}^c}$ of $\mathcal{B}_b$ in $L^\#(\mathfrak{D})$;
(6) the $(\mathfrak{R}^c)$-$\sigma$-strong closure $[\mathcal{B}_b]^\sigma_{\mathfrak{R}^c}$ of $\mathcal{B}_b$ in $L^\#(\mathfrak{D})$.

**Proof.** Clearly we have

$$ [\mathcal{B}_b]^\sigma \subseteq [\mathcal{B}_b]^\omega \subseteq [\mathcal{B}_b]_{\mathfrak{R}^c}. $$

Hence we have only to show that $\mathfrak{R}^c \subseteq [\mathcal{B}_b]^\sigma_{\mathfrak{R}^c}$ and $[\mathcal{B}_b]^\omega \subseteq [\mathcal{B}_b]_{\mathfrak{R}^c}$. For each $\xi_\infty = (\xi_1, \xi_2, \ldots) \in \mathfrak{R}^c$ and $T \in \mathfrak{R}^c$, putting $T_\infty \xi_\infty = (T \xi_1, T \xi_2, \ldots)$, $T_\infty$ is a linear operator on $\mathfrak{R}^c$. It is easily shown that $(\mathfrak{R}^c)_\infty := \{ T_\infty : T \in \mathfrak{R}^c \}$ is a closed symmetric $\#$-algebra on $\mathfrak{R}^c$ under the operations

$$ S_\infty + T_\infty = (S + T)_\infty, \lambda T_\infty = (\lambda T)_\infty, S_\infty T_\infty = (ST)_\infty \text{ and } T^\*= (T^*)_\infty. $$

Suppose $T \in \mathfrak{R}^c$. Then $T_\infty \in (\mathfrak{R}^c)_\infty$. From Proposition 1, $T_\infty \eta((\mathfrak{R}^c)_\infty)_{\mathfrak{R}^c}$. Hence, $T_\infty \in (\mathfrak{R}^c)_\infty$ and it is easily proved that $((\mathfrak{R}^c)_\infty)^* = (\mathfrak{R}^c)_\infty$. Hence, $T_\infty \in (\mathfrak{R}^c)_\infty$ and it follows that $T \in [\mathfrak{R}^c]^\omega_{\mathfrak{R}^c}$. Furthermore, $\mathfrak{R}^c$ is $\sigma$-strongly dense in $\mathfrak{R}^c$, and so $T \in [\mathcal{B}_b]^\sigma_{\mathfrak{R}^c}$. Thus, $\mathfrak{R}^c \subseteq [\mathcal{B}_b]^\sigma_{\mathfrak{R}^c}$. From Proposition 3, $\mathfrak{R}^c$ is weakly closed and it follows that $[\mathcal{B}_b]^\omega \subseteq [\mathcal{B}_b]^\sigma_{\mathfrak{R}^c}$.

**Corollary.** (1) $\mathfrak{R}^c(\mathfrak{D}_0)$ equals the weak closure of $\pi^\omega_2(\mathfrak{D}_0)$ in $L^\#(L_2^\omega(\mathfrak{D}_0))$.
(2) If $\mathfrak{D}_0$ has an identity or if $\mathfrak{D}_0$ is separable, then $\mathfrak{R}(L_2^\omega(\mathfrak{D}_0))$ equals the weak closure of $\pi^\omega_2(\mathfrak{D}_0)$ in $L^\#(L_2^\omega(\mathfrak{D}_0))$.

**References**


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