PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 69, Number 1, April 1978

A COMMUTANT OF AN UNBOUNDED OPERATOR ALGEBRA

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ABSTRACT. A commutant $\mathfrak{A}^c$ and bicommutant $\mathfrak{A}^{cc}$ of an unbounded operator algebra $\mathfrak{A}$ called a #-algebra are defined. The first purpose of this paper is to investigate whether the bicommutant $\mathfrak{A}^{cc}$ of a #-algebra $\mathfrak{A}$ is an $EW^\#$-algebra, as defined in [6], or not. The second purpose is to investigate the relation between $\mathfrak{A}^{cc}$ and topologies on a #-algebra $\mathfrak{A}$.

In this paper let $\mathfrak{D}$ be a pre-Hilbert space with an inner product $(\ |\ )$ and $\mathfrak{h}$ the completion of $\mathfrak{D}$. Let $\mathfrak{L}(\mathfrak{D})$ denote the set of all linear operators on $\mathfrak{D}$ and $\mathfrak{L}^\#(\mathfrak{D})$ the set $\{A \in \mathfrak{L}(\mathfrak{D});\ A^*\mathfrak{D} \subset \mathfrak{D}\}$. Every element A of $\mathfrak{L}^\#(\mathfrak{D})$ is a closable operator on $\mathfrak{h}$ with the domain $\mathfrak{D}$. For each $A \in \mathfrak{L}^\#(\mathfrak{D})$, putting $A^\# = A^*/\mathfrak{D}$ (the restriction of A^* onto $\mathfrak{D}$), the map $A \to A^\#$ is an involution on $\mathfrak{L}^\#(\mathfrak{D})$ and $\mathfrak{L}^\#(\mathfrak{D})$ is an algebra of operators on $\mathfrak{D}$ with the involution #.

If $\mathfrak{A}$ is a #-subalgebra of $\mathfrak{L}^\#(\mathfrak{D})$, then it is called a #-algebra on $\mathfrak{D}$. In particular, $\mathfrak{L}^\#(\mathfrak{D})$ is called a maximal #-algebra on $\mathfrak{D}$. We denote by $\bar{S}$ the smallest closed extension of $S \in \mathfrak{A}$ and by $\bar{\mathfrak{A}}$ the set $\{\bar{S};\ S \in \mathfrak{A}\}$. We set

$$\mathfrak{A}_b = \left\{A \in \mathfrak{A};\ \bar{A} \in \mathfrak{B}(\mathfrak{h})\right\},$$

where $\mathfrak{B}(\mathfrak{h})$ denotes the set of all bounded linear operators on $\mathfrak{h}$, and call it the bounded part of $\mathfrak{A}$. A #-algebra $\mathfrak{A}$ is called pure if $\mathfrak{A} \neq \mathfrak{A}_b$. A #-algebra $\mathfrak{A}$ on $\mathfrak{D}$ is said to be symmetric if $\mathfrak{A}$ has an identity operator I and $(I + A^\# A)^{-1} \in \mathfrak{A}_b$ for all $A \in \mathfrak{A}$. A symmetric #-algebra $\mathfrak{A}$ on $\mathfrak{D}$ is called an $EW^\#$-algebra on $\mathfrak{D}$ over $\overline{\mathfrak{A}_b}$ if $\overline{\mathfrak{A}_b}$ is a von Neumann algebra.

A #-algebra $\mathfrak{A}$ on $\mathfrak{D}$ is said to be closed (resp. selfadjoint) if

$$\mathfrak{D} = \bigcap_{A \in \mathfrak{A}} \mathfrak{D}(\bar{A}) \quad \left(\text{resp. } \mathfrak{D} = \bigcap_{A \in \mathfrak{A}} \mathfrak{D}(A^*)\right).$$

It is easy to show that if $\mathfrak{A}$ is a selfadjoint #-algebra on $\mathfrak{D}$ then it is closed. By [**6**, Proposition 2.6], if $\mathfrak{A}$ is a closed symmetric #-algebra, then it is selfadjoint.

In [**6**] we defined the commutant $\mathfrak{A}'$ of a #-algebra $\mathfrak{A}$ on $\mathfrak{D}$ as follows:

$$\mathfrak{A}' = \left\{C \in \mathfrak{B}(\mathfrak{h});\ (CA\xi|\eta) = (C\xi|A^\#\eta) \text{ for all } A \in \mathfrak{A} \text{ and } \xi, \eta \in \mathfrak{D}\right\}.$$

From [**6**, Proposition 2.8] if $\mathfrak{A}$ is a selfadjoint #-algebra on $\mathfrak{D}$ then $\mathfrak{A}'$ is a von

Received by the editors December 13, 1976.
AMS (MOS) subject classifications (1970). Primary 46L20.

Neumann algebra. Furthermore, for each $C \in \mathfrak{A}'$, $A \in \mathfrak{A}$ and $\xi \in \mathfrak{D}$ we have $C\mathfrak{D} \subset \mathfrak{D}$ and $CA\xi = AC\xi$. We define a new commutant $\mathfrak{A}^c$ and bicommutant $\mathfrak{A}^{cc}$ as follows:

$$\mathfrak{A}^c = \left\{S \in \mathfrak{L}^{\#}(\mathfrak{D});\ SA = AS \text{ for all } A \in \mathfrak{A}\right\},$$

$$\mathfrak{A}^{cc} = \left\{A \in \mathfrak{L}^{\#}(\mathfrak{D});\ SA = AS \text{ for all } S \in \mathfrak{A}^c\right\}.$$

It is immediately shown that $\mathfrak{A}^c$, $\mathfrak{A}^{cc}$ are $\#$-algebras on $\mathfrak{D}$ and $\mathfrak{A}^{cc} \supset \mathfrak{A}$.

Lemma 1. *If $\mathfrak{A}$ is a selfadjoint $\#$-algebra on $\mathfrak{D}$, then*:

(1) *$\overline{T}$ is affiliated with $(\mathfrak{A}_b)'$ (written $\overline{T}\eta(\mathfrak{A}_b)'$) for each $T \in \mathfrak{A}^c$ and $(\mathfrak{A}^c)_b = \mathfrak{A}' \subset (\mathfrak{A}_b)'$*;

(2) *$\overline{A}\eta\mathfrak{A}''$ for each $A \in \mathfrak{A}^{cc}$ and $\overline{\mathfrak{A}_b} \subset \overline{(\mathfrak{A}^{cc})_b} \subset \mathfrak{A}''$.*

Lemma 2. *If $\mathfrak{A}$ is a symmetric $\#$-algebra on $\mathfrak{D}$, then $(\mathfrak{A}_b)' = \mathfrak{A}'$ and $(\mathfrak{A}_b)'' = \mathfrak{A}''$.*

Proof. Clearly, $\mathfrak{A}' \subset (\mathfrak{A}_b)'$. Suppose that $C \in (\mathfrak{A}_b)'$. Let $A \in \mathfrak{A}_h := \{A \in \mathfrak{A};\ A^{\#} = A\}$. Since $\mathfrak{A}$ is symmetric, it is easily shown that $(I + A^2)^{-1}$, $A(I + A^2)^{-1} \in \mathfrak{A}_b$. For each $\xi, \eta \in \mathfrak{D}$ we have

$$\left(CA(I + A^2)^{-1}\xi|\eta\right) = \left(A(I + A^2)^{-1}C\xi|\eta\right) = \left(AC(I + A^2)^{-1}\xi|\eta\right).$$

Since $(I + A^2)^{-1}\mathfrak{D} = \mathfrak{D}$, we get $C \in \mathfrak{A}'$. Thus, $(\mathfrak{A}_b)' = \mathfrak{A}'$.

Proposition 1. *Let $\mathfrak{A}$ be a closed symmetric $\#$-algebra on $\mathfrak{D}$. Then*:

(1) *$\mathfrak{A}^c$ is an $EW^{\#}$-algebra on $\mathfrak{D}$ over $\mathfrak{A}'$*;

(2) *$\mathfrak{A}^{cc} = \{A \in \mathfrak{L}^{\#}(\mathfrak{D});\ \overline{A}\eta\mathfrak{A}'' = (\mathfrak{A}_b)''\}$*;

(3) *if $(\mathfrak{A}_b)''\mathfrak{D} \subset \mathfrak{D}$, then $\mathfrak{A}^{cc}$ is a closed $EW^{\#}$-algebra on $\mathfrak{D}$ over $(\mathfrak{A}_b)''$.*

Proof. From [**6**, Proposition 2.6] $\mathfrak{A}$ is a selfadjoint $\#$-algebra on $\mathfrak{D}$. Hence this follows from Lemma 1.2.

Corollary. *If $\mathfrak{A}$ is a closed $EW^{\#}$-algebra on $\mathfrak{D}$, then $\mathfrak{A}^c$ is an $EW^{\#}$-algebra on $\mathfrak{D}$ over $\mathfrak{A}'$ and $\mathfrak{A}^{cc}$ is a closed $EW^{\#}$-algebra on $\mathfrak{D}$ over $\overline{\mathfrak{A}_b}$.*

In this paper let $\mathcal{D}$ be an unbounded Hilbert algebra over $\mathcal{D}_0$ and $\mathfrak{h}(\mathcal{D}_0)$ the completion of $\mathcal{D}_0$. For the basic definitions and facts of unbounded Hilbert algebras the reader is referred to [**7**], [**8**]. Let $\mathcal{U}_0(\mathcal{D}_0)$ (resp. $\mathcal{V}_0(\mathcal{D}_0)$) be the left (resp. right) von Neumann algebra of the Hilbert algebra $\mathcal{D}_0$. Let π_0 (resp. π_0') be the left (resp. right) regular representation of $\mathcal{D}_0$ and ϕ_0 (resp. ϕ_0') the natural trace on $\mathcal{U}_0(\mathcal{D}_0)^+$ (resp. $\mathcal{V}_0(\mathcal{D}_0)^+$). For each $x \in \mathfrak{h}(\mathcal{D}_0)$ we define $\pi_0(x)$ and $\pi_0'(x)$ by

$$\pi_0(x)\xi = \overline{\pi_0'(\xi)}\,x, \quad \pi_0'(x)\xi = \overline{\pi_0(\xi)}\,x \qquad (\xi \in \mathcal{D}_0).$$

Then $\pi_0(x)$ and $\pi_0'(x)$ are linear operators on $\mathfrak{h}(\mathcal{D}_0)$ with domain $\mathcal{D}_0$ and $\overline{\pi_0(x^*)} = \pi_0(x)^*$, $\overline{\pi_0'(x^*)} = \pi_0'(x)^*$. Putting $(\mathcal{D}_0)_b = \{x \in \mathfrak{h}(\mathcal{D}_0);\ \overline{\pi_0(x)} \in \mathfrak{B}(\mathfrak{h}(\mathcal{D}_0))\}$, $(\mathcal{D}_0)_b$ is a Hilbert algebra containing $\mathcal{D}_0$ and is called the maximal Hilbert algebra of $\mathcal{D}_0$ in $\mathfrak{h}(\mathcal{D}_0)$.

Let $\mathfrak{M}$ (resp. $\mathfrak{M}^+$) be the set of all measurable (resp. positive measurable) operators on $\mathfrak{h}(\mathfrak{D}_0)$ with respect to $\mathfrak{U}_0(\mathfrak{D}_0)$. For each $T \in \mathfrak{M}^+$ we set

$$\mu_0(T) = \sup\left[\phi_0\left(\overline{\pi_0(\xi)}\right); 0 \leqslant \overline{\pi_0(\xi)} \leqslant T, \xi \in (\mathfrak{D}_0)_b^2\right],$$

$$L^p(\phi_0) = \left\{T \in \mathfrak{M}; \|T\|_p := \mu_0(|T|^p)^{1/p} < \infty\right\}, \quad 1 \leqslant p < \infty,$$

$$L^\infty(\phi_0) = \mathfrak{U}_0(\mathfrak{D}_0).$$

We define L_2^ω-spaces with respect to ϕ_0 and $\mathfrak{D}_0$ as follows:

$$L_2^\omega(\phi_0) = \bigcap_{2 \leqslant p < \infty} L^p(\phi_0), \quad L_2^\omega(\mathfrak{D}_0) = \left\{x \in \mathfrak{h}; \overline{\pi_0(x)} \in L_2^\omega(\phi_0)\right\},$$

respectively. By [7, Theorem 3.9] $L_2^\omega(\mathfrak{D}_0)$ is maximal among unbounded Hilbert algebras containing $\mathfrak{D}_0$ and is called the maximal unbounded Hilbert algebra of $\mathfrak{D}_0$. Let π_2^ω (resp. $(\pi')_2^\omega$) be the left (resp. right) regular representation (i.e., $\pi_2^\omega(x)y = xy$, $(\pi')_2^\omega(x)y = yx$, $x, y \in L_2^\omega(\mathfrak{D}_0)$) of $L_2^\omega(\mathfrak{D}_0)$. We set

$$\mathfrak{U}_0(\mathfrak{D}_0)/L_2^\omega(\mathfrak{D}_0) = \{T/L_2^\omega(\mathfrak{D}_0); T \in \mathfrak{U}_0(\mathfrak{D}_0)\},$$

$$\mathfrak{V}_0(\mathfrak{D}_0)/L_2^\omega(\mathfrak{D}_0) = \{T'/L_2^\omega(\mathfrak{D}_0); T' \in \mathfrak{V}_0(\mathfrak{D}_0)\}.$$

Then $\pi_2^\omega(\mathfrak{D})$, $(\pi')_2^\omega(\mathfrak{D})$, $\mathfrak{U}_0(\mathfrak{D}_0)/L_2^\omega(\mathfrak{D}_0)$ and $\mathfrak{V}_0(\mathfrak{D}_0)/L_2^\omega(\mathfrak{D}_0)$ are #-algebras on $L_2^\omega(\mathfrak{D}_0)$. We denote by $\mathfrak{U}(\mathfrak{D})$ (resp. $\mathfrak{V}(\mathfrak{D})$) a #-algebra on $L_2^\omega(\mathfrak{D}_0)$ generated by $\pi_2^\omega(\mathfrak{D})$ and $\mathfrak{U}_0(\mathfrak{D}_0)/L_2^\omega(\mathfrak{D}_0)$ (resp. $(\pi')_2^\omega(\mathfrak{D})$ and $\mathfrak{V}_0(\mathfrak{D}_0)/L_2^\omega(\mathfrak{D}_0)$). Then $\mathfrak{U}(\mathfrak{D})$ (resp. $\mathfrak{V}(\mathfrak{D})$) is an $EW^\#$-algebra on $L_2^\omega(\mathfrak{D}_0)$ over $\mathfrak{U}_0(\mathfrak{D}_0)$ (resp. $\mathfrak{V}_0(\mathfrak{D}_0)$) and is called the left (resp. right) $EW^\#$-algebra of $\mathfrak{D}$.

THEOREM 1. *Suppose that $\mathfrak{h}(\mathfrak{D}_0)$ is not a Hilbert algebra, i.e., $\mathfrak{h}(\mathfrak{D}_0) \neq (\mathfrak{D}_0)_b$. Then*:

(1) *$\pi_2^\omega(\mathfrak{D}_0)^c$ is a pure $EW^\#$-algebra on $L_2^\omega(\mathfrak{D}_0)$ over $\mathfrak{V}_0(\mathfrak{D}_0)$ such that*

$$\pi_2^\omega(\mathfrak{D}_0)^c = \left\{T \in \mathfrak{L}^\#(L_2^\omega(\mathfrak{D}_0)); \bar{T}\eta\mathfrak{V}_0(\mathfrak{D}_0)\right\} \supset \mathfrak{V}(L_2^\omega(\mathfrak{D}_0));$$

(2) *$\pi_2^\omega(\mathfrak{D}_0)^{cc}$ is a pure $EW^\#$-algebra on $L_2^\omega(\mathfrak{D}_0)$ over $\mathfrak{U}_0(\mathfrak{D}_0)$ such that*

$$\pi_2^\omega(\mathfrak{D}_0)^{cc} = \left\{A \in \mathfrak{L}^\#(L_2^\omega(\mathfrak{D}_0)); \bar{A}\eta\mathfrak{U}_0(\mathfrak{D}_0)\right\} \supset \mathfrak{U}(L_2^\omega(\mathfrak{D}_0)).$$

PROOF. (1) It is easily proved that $\pi_2^\omega(\mathfrak{D}_0)^c = \{T \in \mathfrak{L}^\#(L_2^\omega(\mathfrak{D}_0)); \bar{T}\eta\mathfrak{V}_0(\mathfrak{D}_0)\}$. Since $\mathfrak{V}_0(\mathfrak{D}_0)L_2^\omega(\mathfrak{D}_0) \subset L_2^\omega(\mathfrak{D}_0)$ and $(\pi')_2^\omega(L_2^\omega(\mathfrak{D}_0))L_2^\omega(\mathfrak{D}_0) \subset L_2^\omega(\mathfrak{D}_0)$, $\mathfrak{V}_0(\mathfrak{D}_0)/L_2^\omega(\mathfrak{D}_0)$ and $(\pi')_2^\omega(L_2^\omega(\mathfrak{D}_0))$ are contained in $\pi_2^\omega(\mathfrak{D}_0)^c$. Hence $\mathfrak{V}(L_2^\omega(\mathfrak{D}_0)) \subset \pi_2^\omega(\mathfrak{D}_0)^c$. Furthermore,

$$\mathfrak{V}_0(\mathfrak{D}_0) \supset \overline{\left(\pi_2^\omega(\mathfrak{D}_0)^c\right)_b} \supset \overline{\mathfrak{V}(\mathfrak{D})_b} = \mathfrak{V}_0(\mathfrak{D}_0).$$

Thus $\pi_2^\omega(\mathfrak{D}_0)^c$ is an $EW^\#$-algebra on $L_2^\omega(\mathfrak{D}_0)$ over $\mathfrak{V}_0(\mathfrak{D}_0)$ containing $\mathfrak{V}(L_2^\omega(\mathfrak{D}_0))$. By [**8**, Theorem 3.4], $\mathfrak{V}(L_2^\omega(\mathfrak{D}_0))$ is pure and it follows that $\pi_2^\omega(\mathfrak{D}_0)^c$ is pure.

(2) This is proved in the same way as (1).

COROLLARY. *If $\mathfrak{D}$ is pure, then*

$$\pi_2^\omega(\mathfrak{D}_0)^c = \pi_2^\omega(\mathfrak{D})^c = \pi_2^\omega(L_2^\omega(\mathfrak{D}_0))^c$$

and

$$\pi_2^\omega(\mathfrak{D}_0)^{cc} = \pi_2^\omega(\mathfrak{D})^{cc} = \pi_2^\omega(L_2^\omega(\mathfrak{D}_0))^{cc}.$$

Furthermore, $\pi_2^\omega(\mathfrak{D}_0)^c$ *(resp.* $\pi_2^\omega(\mathfrak{D}_0)^{cc}$*) is maximal among* $EW^{\#}$*-algebras on* $L_2^\omega(\mathfrak{D}_0)$ *over* $\mathfrak{V}_0(\mathfrak{D}_0)$ *(resp.* $\mathfrak{U}_0(\mathfrak{D}_0)$*).*

We call $\pi_2^\omega(\mathfrak{D}_0)^{cc}$ (resp. $\pi_2^\omega(\mathfrak{D}_0)^c$) the maximal left (resp. right) $EW^{\#}$-algebra of $\mathfrak{D}_0$ and denote it by $\mathfrak{M}^l(\mathfrak{D}_0)$ (resp. $\mathfrak{M}^r(\mathfrak{D}_0)$).

PROPOSITION 2. *Let* $\mathfrak{M}(\phi_0)$ *(resp.* $\mathfrak{M}(\phi_0')$*) be the set of all* ϕ_0*-measurable (resp.* ϕ_0'*-measurable) operators. Then*:

(1) $$\mathfrak{U}(L_2^\omega(\mathfrak{D}_0)) = \left\{A \in \mathfrak{L}^{\#}(L_2^\omega(\mathfrak{D}_0));\ \bar{A} \in \mathfrak{M}(\phi_0)\right\};$$

(2) $$\mathfrak{V}(L_2^\omega(\mathfrak{D}_0)) = \left\{T \in \mathfrak{L}^{\#}(L_2^\omega(\mathfrak{D}_0));\ \bar{T} \in \mathfrak{M}(\phi_0')\right\}.$$

PROOF. (1) Suppose that $A \in \mathfrak{L}^{\#}(L_2^\omega(\mathfrak{D}_0))$ and $\bar{A} \in \mathfrak{M}(\phi_0)$. Let $\bar{A} = U|\bar{A}|$ be the polar decomposition of $\bar{A}$ and $|\bar{A}| = \int_0^\infty \lambda\, dE(\lambda)$ the spectral resolution of $|\bar{A}|$. Since $\bar{A}$ is ϕ_0-measurable, $|\bar{A}|$ is ϕ_0-measurable, and it follows that $E(\lambda_0)^\perp := I - E(\lambda_0) \in \overline{\pi_2^\omega((\mathfrak{D}_0)_b)}$ for some $\lambda_0 > 0$, i.e., there exists an element e_{λ_0} of $(\mathfrak{D}_0)_b$ such that $E(\lambda_0)^\perp = \overline{\pi_2^\omega(e_{\lambda_0})}$. Putting

$|A| = |\bar{A}|/L_2^\omega(\mathfrak{D}_0)$,

$A_0 = [\int_0^{\lambda_0} \lambda\, dE(\lambda)]/L_2^\omega(\mathfrak{D}_0)$ and $U_0 = U/L_2^\omega(\mathfrak{D}_0)$,

$|A| \in \mathfrak{L}^{\#}(L_2^\omega(\mathfrak{D}_0))$ and $A_0, U_0 \in \mathfrak{U}_0(\mathfrak{D}_0)/L_2^\omega(\mathfrak{D}_0)$.

Furthermore,

$$A = U_0|A| = U_0A_0 + \pi_2^\omega(U_0|A|e_{\lambda_0}) \in \mathfrak{U}_0(\mathfrak{D}_0)/L_2^\omega(\mathfrak{D}_0) + \pi_2^\omega(L_2^\omega(\mathfrak{D}_0)).$$

Hence, $A \in \mathfrak{U}(L_2^\omega(\mathfrak{D}_0))$. Thus, $\{A \in \mathfrak{L}^{\#}(L_2^\omega(\mathfrak{D}_0));\ \bar{A} \in \mathfrak{M}(\phi_0)\} \subset \mathfrak{U}(L_2^\omega(\mathfrak{D}_0))$. The reverse inclusion is obvious.

(2) This is proved in the same way as (1).

THEOREM 2. *If* $\mathfrak{D}_0$ *has an identity or* $\mathfrak{h}(\mathfrak{D}_0)$ *is separable, then* $\mathfrak{M}^l(\mathfrak{D}_0) = \mathfrak{U}(L_2^\omega(\mathfrak{D}_0))$ *and* $\mathfrak{M}^r(\mathfrak{D}_0) = \mathfrak{V}(L_2^\omega(\mathfrak{D}_0))$.

PROOF. If $\mathfrak{D}_0$ has an identity, then this is easily proved. Suppose that $\mathfrak{h}(\mathfrak{D}_0)$ is separable. P. G. Dixon [**4**, Theorem 5.3] has proved that each EW^*-algebra $\mathfrak{A}$ over a von Neumann algebra $\mathfrak{A}_b$ is contained in the algebra $\mathfrak{M}(\mathfrak{A}_b)$ of all measurable operators with respect to $\mathfrak{A}_b$. From Theorem 1, $\overline{\mathfrak{M}^l(\mathfrak{D}_0)}$ is an EW^*-algebra over $\mathfrak{U}_0(\mathfrak{D}_0)$. Hence, $\overline{\mathfrak{M}^l(\mathfrak{D}_0)} \subset \mathfrak{M}(\mathfrak{U}_0(\mathfrak{D}_0))$. Suppose that $A \in \mathfrak{M}^l(\mathfrak{D}_0)$. Let $\bar{A} = U|\bar{A}|$ be the polar decomposition of $\bar{A}$ and $|\bar{A}| = \int_0^\infty \lambda\, dE(\lambda)$ the spectral resolution of $|\bar{A}|$. Since $\bar{A}$ is measurable, $E_0 := E(\lambda_0)^\perp$ is a finite projection for some $\lambda_0 > 0$ and it follows that $\mathfrak{U}_0(\mathfrak{D}_0)_{E_0}$ is a finite von Neumann algebra. Furthermore, since $\mathfrak{h}(\mathfrak{D}_0)$ is separable, $\mathfrak{U}_0(\mathfrak{D}_0)_{E_0}$ is σ-finite. From [3, §6, Proposition 9], there exists a faithful normal finite trace χ_0 on $\mathfrak{U}_0(\mathfrak{D}_0)_{E_0}^+$ and it follows that there exists an isomorphism Ψ of $\mathfrak{U}_0(\mathfrak{D}_0)_{E_0}$ onto a standard von Neumann algebra $\mathfrak{A}_0$ such that $\chi_0(T) =$

$\chi(\Psi(T))$ for every $T \in \mathcal{U}_0(\mathcal{D}_0)^+_{E_0}$, where χ denotes the natural trace on $\mathfrak{A}_0^+$ [3, §6, Theorem 2]. We set

$$(\phi_0)_{E_0}(T) = \phi_0(TE_0), \qquad T \in \mathcal{U}_0(\mathcal{D}_0)^+_{E_0}.$$

Then it is easily proved that $(\phi_0)_{E_0}$ is a faithful normal semifinite trace on $\mathcal{U}_0(\mathcal{D}_0)^+_{E_0}$. Hence, from [3, §6, Theorem 2] there exists an isomorphism Φ of $\mathcal{U}_0(\mathcal{D}_0)_{E_0}$ onto a standard von Neumann algebra $\mathfrak{B}_0$ such that $(\phi_0)_{E_0}(T) = \phi(\Phi(T))$ for every $T \in \mathcal{U}_0(\mathcal{D}_0)^+_{E_0}$, where ϕ denotes the natural trace on $\mathfrak{B}_0^+$. Then the standard von Neumann algebras $\mathfrak{A}_0$ and $\mathfrak{B}_0$ are isomorphic. From [3, §6, Theorem 4] $\mathfrak{A}_0$ and $\mathfrak{B}_0$ are spatially isomorphic. Since $\chi(T) < \infty$ for all $T \in \mathfrak{A}_0^+$, $\phi(T) < \infty$ for all $T \in \mathfrak{B}_0^+$. Hence, $\phi_0(E(\lambda_0)^\perp) = \phi(\Phi(I)) < \infty$. Thus we can show that if S is measurable with respect to $\mathcal{U}_0(\mathcal{D}_0)$ then S is ϕ_0-measurable. Hence, Theorem 2 follows from Proposition 2.

Next we shall investigate the relation between the commutants $\mathfrak{A}^c$, $\mathfrak{A}^{cc}$ of a #-algebra $\mathfrak{A}$ on $\mathfrak{D}$ and topologies on $\mathfrak{A}$. The locally convex topology induced by seminorms: $P_{\xi,\eta}(T) := |(T\xi|\eta)|$ $(\xi, \eta \in \mathfrak{D})$ is called the weak topology on $\mathfrak{A}$. Let $\mathfrak{B}$ be a #-algebra on $\mathfrak{D}$ containing $\mathfrak{A}$. We set

$$\mathcal{D}_\infty(\mathfrak{B}) = \Big\{ \xi_\infty = (\xi_1, \xi_2, \dots, \xi_n, \dots);\ \xi_n \in \mathfrak{D}\ (n = 1, 2, \dots) \text{ and } \sum_{n=1}^{\infty} \|T\xi_n\|_2^2 < \infty \text{ for all } T \in \mathfrak{B} \Big\}.$$

For each $\xi_\infty = (\xi_1, \xi_2, \dots, \xi_n, \dots)$ and $\eta_\infty = (\eta_1, \eta_2, \dots, \eta_n, \dots)$ in $\mathcal{D}_\infty(\mathfrak{B})$, $P_{\xi_\infty,\eta_\infty}(T) := |\sum_{n=1}^\infty (T\xi_n|\eta_n)|$ $(T \in \mathfrak{A})$ is a seminorm on $\mathfrak{A}$. The locally convex topology on $\mathfrak{A}$ induced by the family $\{P_{\xi_\infty,\eta_\infty}(\ );\ \xi_\infty, \eta_\infty \in \mathcal{D}_\infty(\mathfrak{B})\}$ of the seminorms is called the $(\mathfrak{B})$-σ-weak topology on $\mathfrak{A}$. In particular, the $(\mathfrak{L}^\#(\mathfrak{D}))$-$\sigma$-weak topology on $\mathfrak{A}$ is simply called the σ-weak topology on $\mathfrak{A}$. It is easy to check that $\mathfrak{A}$ is a locally convex $*$-algebra under the involution # and weak topology (or, $(\mathfrak{B})$-σ-weak topology). The strong topology is the locally convex topology induced by seminorms: $P_\xi(T) = \|T\xi\|$ $(\xi \in \mathfrak{D})$. For each $\xi_\infty = (\xi_1, \xi_2, \dots, \xi_n, \dots) \in \mathcal{D}_\infty(\mathfrak{B})$, $P_{\xi_\infty}(T) := (\sum_{n=1}^\infty \|T\xi_n\|_2^2)^{1/2}$ $(T \in \mathfrak{A})$ is a seminorm on $\mathfrak{A}$. The locally convex topology induced by the seminorms $\{P_{\xi_\infty}(\ );\ \xi_\infty \in \mathcal{D}_\infty(\mathfrak{B})\}$ is called the $(\mathfrak{B})$-σ-strong topology on $\mathfrak{A}$. In particular, the $(\mathfrak{L}^\#(\mathfrak{D}))$-$\sigma$-strong topology on $\mathfrak{A}$ is simply called the σ-strong topology on $\mathfrak{A}$.

PROPOSITION 3. *If $\mathfrak{A}$ is a #-algebra on $\mathfrak{D}$, then $\mathfrak{A}^c$ and $\mathfrak{A}^{cc}$ are closed in $\mathfrak{L}^\#(\mathfrak{D})$ under the weak topology.*

COROLLARY. *$\mathfrak{M}^l(\mathcal{D}_0)$ and $\mathfrak{M}^r(\mathcal{D}_0)$ are closed in $\mathfrak{L}^\#(L_2^\omega(\mathcal{D}_0))$ under the weak topology.*

THEOREM 3. *If $\mathfrak{A}$ is a closed symmetric #-algebra on $\mathfrak{D}$, then the following algebras, (1) $\sim$ (6), coincide with $\mathfrak{A}^{cc}$:*

(1) *the weak closure $[\mathfrak{A}_b]^w$ of $\mathfrak{A}_b$ in $\mathfrak{L}^\#(\mathfrak{D})$;*

(2) *the strong closure $[\mathfrak{A}_b]^s$ of $\mathfrak{A}_b$ in $\mathfrak{L}^\#(\mathfrak{D})$;*

(3) *the σ-weak closure* $[\mathfrak{A}_b]^{\sigma\omega}$ *of* $\mathfrak{A}_b$ *in* $\mathfrak{L}^{\#}(\mathfrak{D})$;
(4) *the σ-strong closure* $[\mathfrak{A}_b]^{\sigma s}$ *of* $\mathfrak{A}_b$ *in* $\mathfrak{L}^{\#}(\mathfrak{D})$;
(5) *the* $(\mathfrak{A}^{cc})$*-σ-weak closure* $[\mathfrak{A}_b]^{\mathfrak{A}^{cc}-\sigma\omega}$ *of* $\mathfrak{A}_b$ *in* $\mathfrak{L}^{\#}(\mathfrak{D})$;
(6) *the* $(\mathfrak{A}^{cc})$*-σ-strong closure* $[\mathfrak{A}_b]^{\mathfrak{A}^{cc}-\sigma s}$ *of* $\mathfrak{A}_b$ *in* $\mathfrak{L}^{\#}(\mathfrak{D})$.

PROOF. Clearly we have

$$\begin{array}{ccc} [\mathfrak{A}_b]^{s} & \subset & [\mathfrak{A}_b]^{\omega} \\ \cup & & \cup \\ [\mathfrak{A}_b]^{\sigma s} & \subset & [\mathfrak{A}_b]^{\sigma\omega} \\ \cup & & \cup \\ [\mathfrak{A}_b]^{\mathfrak{A}^{cc}-\sigma s} & \subset & [\mathfrak{A}_b]^{\mathfrak{A}^{cc}-\sigma\omega}. \end{array}$$

Hence we have only to show that $\mathfrak{A}^{cc} \subset [\mathfrak{A}_b]^{\mathfrak{A}^{cc}-\sigma s}$ and $[\mathfrak{A}_b]^{\omega} \subset \mathfrak{A}^{cc}$. For each $\xi_\infty = (\xi_1, \xi_2, \dots) \in \mathfrak{D}_\infty(\mathfrak{A}^{cc})$ and $T \in \mathfrak{A}^{cc}$, putting $T_\infty \xi_\infty = (T\xi_1, T\xi_2, \dots)$, T_∞ is a linear operator on $\mathfrak{D}_\infty(\mathfrak{A}^{cc})$. It is easily shown that $(\mathfrak{A}^{cc})_\infty := \{T_\infty; T \in \mathfrak{A}^{cc}\}$ is a closed symmetric #-algebra on $\mathfrak{D}_\infty(\mathfrak{A}^{cc})$ under the operations $S_\infty + T_\infty = (S+T)_\infty$, $\lambda T_\infty = (\lambda T)_\infty$, $S_\infty T_\infty = (ST)_\infty$ and $T_\infty^{\#} = (T^{\#})_\infty$. Suppose $T \in \mathfrak{A}^{cc}$. Then $T_\infty \in (\mathfrak{A}^{cc})_\infty$. From Proposition 1, $\overline{T}_\infty \eta ((\mathfrak{A}^{cc})_\infty)_b''$. Hence, $T_\infty \in [((\mathfrak{A}^{cc})_\infty)_b'']^{s}$. It is easily proved that $((\mathfrak{A}^{cc})_\infty)_b'' = (\mathfrak{A}_b'')_\infty$. Hence, $T_\infty \in [(\mathfrak{A}_b'')_\infty]^{s}$ and it follows that $T \in [\mathfrak{A}_b'']^{\mathfrak{A}^{cc}-\sigma s}$. Furthermore, $\mathfrak{A}_b$ is σ-strongly dense in $\mathfrak{A}_b''$, and so $T \in [\mathfrak{A}_b]^{\mathfrak{A}^{cc}-\sigma s}$. Thus, $\mathfrak{A}^{cc} \subset [\mathfrak{A}_b]^{\mathfrak{A}^{cc}-\sigma s}$. From Proposition 3, $\mathfrak{A}^{cc}$ is weakly closed and it follows that $[\mathfrak{A}_b]^{\omega} \subset \mathfrak{A}^{cc}$.

COROLLARY. (1) $\mathfrak{M}^{l}(\mathfrak{D}_0)$ *equals the weak closure of* $\pi_2^{\omega}(\mathfrak{D}_0)$ *in* $\mathfrak{L}^{\#}(L_2^{\omega}(\mathfrak{D}_0))$.
(2) *If* $\mathfrak{D}_0$ *has an identity or* $\mathfrak{h}(\mathfrak{D}_0)$ *is separable, then* $\mathfrak{U}(L_2^{\omega}(\mathfrak{D}_0))$ *equals the weak closure of* $\pi_2^{\omega}(\mathfrak{D}_0)$ *in* $\mathfrak{L}^{\#}(L_2^{\omega}(\mathfrak{D}_0))$.

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