ABSOLUTELY CONTINUOUS AND LOCALLY
QUASI-ININVARIANT MEASURES
ON LOCALLY COMPACT SEMIGROPS

HENRY A. M. DZINOTIWEYI

Abstract. Let $S$ be a locally compact compact subsemigroup of $S$. It is shown that, for a large class of semigroups $S$, if $T$ is 'reasonably large', the absolutely continuous measures on $T$ extend to measures whose translates by some points of $S$ are absolutely continuous on $S$. As a consequence we prove the absolute continuity of translates of some measures related to a locally quasi-invariant measure.

1. Introduction. Throughout this paper, let $S$ denote a locally compact jointly continuous topological semigroup, $M(S)$ the set of all bounded complex-valued Radon measures on $S$ and $C_0(S)$ the set of all continuous functions on $S$ which vanish at infinity. Identifying $M(S)$ as the first dual of $C_0(S)$, it is well known that $M(S)$ is a Banach algebra under the usual norm, $|||$, and convolution multiplication given by

$$
\mu * \nu(f) = \int \int f(xy) \, d\mu(x) \, d\nu(x) \quad (\mu, \nu \in M(S); f \in C_0(S)).
$$

In the papers [1], [2], [3] and [7] certain subspaces of the algebra $M(S)$ are studied—the purpose being to find an analogue of $L^1(G)$ for a locally compact group $G$. Let $M_a(S)$ ($M_b(S)$) be the set of all measures $\mu$ in $M(S)$ such that the mapping $x \mapsto x * |\mu| (x \mapsto |\mu| * x)$ of $S$ into $M(S)$ is weakly continuous, where $3_c$ denotes the point mass at $x$ and $|\mu|$ the total variation of $\mu$. Then taking $M_a(S)$ to be $M_a(S) \cap M_b(S)$ we have the set studied in [2], [3] and [9]. In particular, from [2, Lemma 2.4 and Theorem 3.2] and [3, Theorem 2.6] we note that $M_a(S)$ is an $L$-ideal of $M(S)$ (i.e. $M_a(S)$ is a norm closed linear subspace of $M(S)$ such that for any $\nu, \mu \in M(S)$ we have (i) if $\nu \ll |\mu|$ and $\mu \in M_a(S)$ then $\nu \in M_a(S)$ and (ii) if $\nu \in M_a(S)$ then $\nu * \mu, \mu * \nu \in M_a(S)$). Further, when $S$ is a group then $M_a(S)$ (or $M_b(S)$) can be identified with the usual group algebra $L^1(S)$ (see e.g. [6, 19.27 and 20.31]). A different approach is made in [7], where it is shown that if $m$ is a positive measure on $S$ such that $m(x^{-1}E) = m(Ex^{-1}) = 0$ whenever $m(E) = 0$ for all Borel $E \subset S$ and $x \in S$, then taking $L(S, m) = \{\nu \in M(S): \nu \ll m\}$, $L(S, m)$ has some
properties related to those of the usual group algebra $L^1(G)$ of a locally compact group $G$. It is natural to ask whether there is any relationship between $L(S, m)$ and $M_\alpha(S)$. We will give a partial answer to this problem in Theorem 3.7. In §4 we will give an example of a semigroup $S$ equal to the support of a measure $m$ with the property (studied in [7] and) mentioned above but with $M_a(S)$ equal to zero.

Further, it is of interest to know the relationship between $M_a(T)$ and $M_a(S)$, when $T$ is a locally compact subsemigroup of $S$. For example for those subsemigroups $T$ of some semigroups $S$ for which we are able to show that $M_a(T) = \{ \nu \in M_a(S) : |\nu|(S \setminus T) = 0 \}$ (cf. Corollary 3.4) we have for instance: If $S$ is commutative and equal to the carrier space of $M_a(S)$, then the bounded multiplicative linear functionals on $\{ \nu \in M_a(S) : |\nu|(S \setminus T) = 0 \}$ are in one-to-one correspondence with the set of all continuous semicharacters on $T$. This follows easily from [3, Theorem 4.4] and one can easily note that [8, Theorem 13] is a special case of this result.

However, when $S$ is not cancellative such subsemigroups $T$ for which $M_a(T) = \{ \nu \in M_a(S) : |\nu|(S \setminus T) = 0 \}$ are not easy to come by (see e.g. Remark 3.6). In that case we obtain a generalization of such an identification in Theorem 3.2.

We begin by collecting together some results and introducing further notation, in §2, which will be needed in §§3 and 4.

2. Preliminaries. For any sets $A$, $B \subseteq S$ and $x \in S$, let $AB = \{ xy : x \in A, y \in B \}$, $A^{-1}B = \{ y \in S : ay \in B \text{ for some } a \in A \}$, $A^{-1}x = A^{-1}\{ x \}$ and similarly define $AB^{-1}$, $Bx^{-1}$ and $xA^{-1}$. We say a measure $\mu$ is absolutely continuous if $\mu \in M_a(S)$. The measure $x * \mu$ is the left translate of $\mu$ by $x \in S$ ($\mu \in M(S)$). For any regular Borel measure $m$ on $S$, let $L_m = \{ x \in S : |m|(B) = 0 \text{ implies } m(x^{-1}B) = 0 \text{ for all Borel } B \subseteq S \}$, $\text{supp}(m) = \{ x \in S : |m|(X) > 0 \text{ whenever } X \text{ is an open neighbourhood of } x \}$ and if $E \subseteq S$ is a Borel set we define the measure $m|_E$ by $m|_E(B) = m(B \cap E)$ for all Borel $B \subseteq S$. For $\mathcal{E} \subseteq M(S)$ and Borel $E \subseteq S$ we let $\mathcal{E}|_E = \{ \mu \in \mathcal{E} : |\mu|(S \setminus E) = 0 \}$. For any $E \subseteq S$, let $d(E) = \{ x \in S : x \in \text{supp}(\mu|_E) \text{ for some Borel } E \subseteq S \}$.

Following [2] and [3] we will say $S$ is a foundation semigroup if $d(S) = S$, and following [4] we will say a regular Borel measure $m$ is (left) locally quasi-invariant if $d(L_m) \neq \emptyset$.

Whenever $S$ has an identity element 1, $S_1$ will denote the set of all $x \in S$ such that $X^{-1}x \cap xX^{-1}$ is a neighbourhood of 1, whenever $X$ is a neighbourhood of $x$. This set $S_1$ plays a very significant role whenever $S$ is a foundation semigroup with an identity element 1 (see e.g. [4] and [9]). In particular, from [9, 6.9 and 7.8] we have the following:

2.1 Lemma. If $S$ is a foundation semigroup with an identity element 1, then $S_1$ is an ideal of $S$ (i.e. $xy, yx \in S_1$ for all $x \in S_1$ and $y \in S$) and $d(S \setminus S_1) = 0$ (i.e. $M_a(S)$ is ‘concentrated’ on $S_1$).
The next result was proved by us in [4].

2.2 Proposition. Let $S$ be a foundation semigroup with identity element 1 and $\mu \in \mathcal{M}(S)$. Then

(i) $x \cdot \mu \in M_a(S)$, for all $x \in S(d(L_\mu) \cap L_\mu)$;

(ii) if $F$ is a subsemigroup containing a countable subset $\mathcal{O}_\mu$ such that if $x \cdot |\mu|(B) = 0 \ (x \in \mathcal{O}_\mu)$ then $x \cdot |\mu|(B) = 0 \ (x \in F)$, we have $x \cdot |\mu| \in M_a(S) \ (x \in S(d(F) \cap F)F)$.

We will need the next lemma in Theorem 3.2.

2.3 Lemma. If $T$ is a locally compact subsemigroup of $S$, there is an isometric algebra isomorphism $\theta: \mathcal{M}(S)_{\mid T} \to \mathcal{M}(T)$ given by $\theta(\mu)(B) = \mu(B \cap T)$ for all Borel $B \subseteq T$ and $\mu \in \mathcal{M}(S)_{\mid T}$.

Proof. Since $T$ is a locally compact subset of $S$, $T$ is the intersection of an open subset and a closed subset of $S$ (see e.g. [11, 17.4]). So $T$ is a Borel subset of $S$. It is not hard to observe that the proof of [12, Theorem 3.4] contains the proof of our lemma. \(\square\)

Finally for any $E \subseteq S$, let $M_a^l(S, E) = \{ \mu \in \mathcal{M}(S): x \cdot |\mu| \in M_a(S) \text{ for some } x \in E \}$ and note that $M_a^l(S, E)$ is a subalgebra of $\mathcal{M}(S)$ such that if $\nu \in \mathcal{M}(S)$, $\mu \in M_a^l(S, E)$ and $\nu \ll |\mu|$ then $\nu \in M_a^l(S, E)$. (This follows from the fact that $M_a(S)$ is an $L$-ideal as pointed out in the introduction.) Further if $S$ is a group and $E \neq \emptyset$, then evidently $M_a^l(S, E) = M_a(S)$.

3. Main results. Throughout this section $S$ is a foundation semigroup with an identity element 1 and $T$ is a locally compact subsemigroup of $S$. We will only give proofs for the left-handed case, from which the right-handed and two-sided cases should be clear.

3.1 Lemma. Let $E$ be a subset of $S$ with $d(E)$ nonempty and $\mu \in \mathcal{M}(S)$. If

\[ \{ x \cdot |\mu|: x \in E^n \} \text{ is a relatively weak compact subset of } \mathcal{M}(S) \ (n \in \omega) \text{ and } F = \bigcup_{n=1}^{\infty} E^n, \]

then $x \cdot |\mu| \in M_a(S)$ \ (\(x \in S(d(F) \cap F)F\)).

Here $E^{n+1} = EE^n \ (n \in \omega)$ and $E^1 = E$.

Proof. By [5, Théorème 2], the relative weak compactness of $\{ x \cdot |\mu|: x \in E^n \}$ implies that $\{ x \cdot |\mu|: x \in E^n \}$ satisfies item (b) of the theorem of [10]. From [10, pp. 139 and 140] it follows that there is a countable set $A_n \subseteq E^n$ such that $x \cdot |\mu|(B) = 0 \ (x \in A_n)$ implies $x \cdot |\mu|(B) = 0 \ (x \in E^n)$.

Let $\mathcal{O}_\mu = \bigcup_{n=1}^{\infty} A_n$ and observe that $x \cdot |\mu|(B) = 0 \ (x \in \mathcal{O}_\mu)$ implies $x \cdot |\mu|(B) = 0 \ (x \in F)$. By Proposition 2.2(ii) we are done.

3.2 Theorem. Suppose $d(T) \neq \emptyset$. Then $M_a^l(T, T \cap S_1)$ is isometrically algebra isomorphic to $M_a^l(S, T \cap S_1)|_{\mathcal{T}} = \{ \nu \in M_a^l(S, T \cap S_1): |\nu|(S \setminus T) = 0 \}$.

Proof. Let $\theta$ be as stated in Lemma 2.3. Let $\nu \in M_a^l(S, S_1 \cap T)|_{\mathcal{T}}$. There
is \( x \in T \cap S_1 \) with \( \bar{x} \cdot |v| \in M_a(S) \). By \([2, 2.4 \text{ and } 3.2]\), \( (\bar{x} \cdot |v|)|_T \in M_a(T) \). Consequently \( \theta((\bar{x} \cdot |v|)|_T) \in M_a(T) \). Since \( x \in T \), by the definition of \( \theta \), we get

\[
\theta((\bar{x} \cdot |v|)|_T) = \bar{x} \cdot \theta(|v|) = \bar{x} \cdot \theta(|v|).
\]

Thus \( \theta(v) \in M_a(T, T \cap S_1) \).

On the other hand suppose \( v' \in M_a(T, T \cap S_1) \). There is a unique \( v \in M(S)|_T \), such that \( \theta(v) = v' \), by Lemma 2.3. There is \( x_0 \in T \cap S_1 \) such that \( \bar{x}_0 \cdot |v'| \in M_a(T) \). Since \( d(T) \neq \emptyset \), there is a measure \( \eta \in M_a(S) \) such that \( \eta(T) > 0 \). By the regularity of \( \eta \), we can find a compact subset \( E \) contained in \( T \), such that \( \eta(E) > 0 \). Thus \( d(E) \neq \emptyset \). Observe that \( E^n \) is a compact subset of \( T (n \in \omega) \). So, by our definition of \( M_a(T) \), \( \{ \bar{y} \cdot (\bar{x}_0 \cdot |v'|): y \in E^n \} \) is a weakly compact subset of \( M(T) (n \in \omega) \). For all \( x, y \in T \) we have

\[
\bar{y} \cdot (\bar{x}_0 \cdot |v'|) = \bar{y} \cdot \bar{x}_0 \cdot \theta(|v|) = \theta(\bar{y} \cdot (\bar{x}_0 \cdot |v'|)).
\]

It follows that \( \{ (\bar{y} \cdot (\bar{x}_0 \cdot |v'|)|_T: y \in E^n \} \) is a weakly compact subset of \( M(T) \) and hence of \( M(S) \), by Hahn-Banach Theorem. But since \( |v|(S \setminus T) = 0 \), we have

\[
(\bar{y} \cdot (\bar{x}_0 \cdot |v'|)|_T = \bar{y} \cdot (\bar{x}_0 \cdot |v'|)(B)
\]

for every Borel \( B \subset S \) and \( y \in E^n \). It follows that \( \{ \bar{y} \cdot (\bar{x}_0 \cdot |v'|): y \in E^n \} \) is a weakly compact subset of \( M(S) (n \in \omega) \). By Lemma 3.1 we then get

\[
\bar{x} \cdot |v| \in M_a(S) \quad (x \in S_1 (d(F) \cap F) Fx_0),
\]

where \( F = \bigcup_{n=1}^{\infty} E^n \). Since \( d(S \setminus S_1) = \emptyset \) and \( d(T) \neq \emptyset \), we have \( S_1 \cap T \neq \emptyset \). Since \( T \) is a subsemigroup and \( S_1 \) is an ideal of \( S \),

\[
(S_1 \cap T)(d(F) \cap F) Fx_0 \subseteq T \cap S_1.
\]

Consequently \( v \in M_a(S, T \cap S_1) \).

It is now evident from Lemma 2.3 that the restriction of \( \theta \) to \( M_a(S, T \cap S_1)|_T \) is the required isometric algebra isomorphism.

3.3 Corollary. If \( d(T) \neq \emptyset \), then \( M_a(T) \subseteq M_a(S, S_1) \).

3.4 Corollary. Suppose \( S \) is cancellative (i.e. \( xy = xz \) or \( yx = zx \) implies \( y = z \) \( (x, y, z \in S) \)) and \( d(T) \neq \emptyset \). Then

\[
M_a(T) = \{ v \in M_a(S): |v|(S \setminus T) = 0 \}.
\]

Proof. Evidently \( M_a(S)|_T \subseteq M_a(T) \) so we only need to show that \( M_a(T) \subseteq M_a(S)|_T \). Let \( v \in M_a(T) \), \( K \) a compact subset of \( S \) and suppose \( x_n \to x \) in \( S \). By Theorem 3.2, there is \( x_0 \in S_1 \) with \( \bar{x}_0 \cdot |v| \in M_a(S) \). Hence
\[ |\nu|(Kx_0^{-1}) - |\nu|(Kx^{-1})| = |\nu|(x_0^{-1}(x_0K)x_0^{-1}) - |\nu|(x_0^{-1}(x_0K)x^{-1})| \]

by cancellation

\[ = |x_0 \cdot |\nu|((x_0K)x_0^{-1}) - x_0 \cdot |\nu|(x_0K)x^{-1}| \to 0. \]

This implies that \( \nu \in M_a'(S) \), by \([2, \text{Lemma 2.4}]\). Similarly \( \nu \in M_a'^{\prime}(S) \) and hence \( \nu \in M_a(S)' \).

In particular, we have the following improvement of \([1, \text{Proposition 1.16(iii)}]\):

3.5 Corollary. Let \( S \) be a group with a Haar measure \( m \). If \( m(T) \neq 0 \), then \( M_a(T) = \{ \nu \in M(S): \nu \ll m \text{ and } |\nu|(S \setminus T) = 0 \} \) (cf. \([9, \text{Theorem 4.11}]\)).

Proof. This follows from Corollary 3.4 and the fact that \( M_a(S) = L(S, m) \) (see \([6, 19.27 \text{ and } 20.31]\)).

3.6 Remark. In general no identification of \( M_a(T) \) and \( M_a(S)' \) can be achieved even when \( S \) is compact commutative and \( d(T) = T \).

For example consider \( S = [0, 1] \) with the usual topology and the operation \( xy = \min(x + y, 1) \). This example is borrowed from \([9]\). \( S \) is evidently a foundation semigroup. Take \( T = [\frac{1}{2}, 1] \) and observe that \( M_a(T) = M(T) \).

However, for all \( x \in [\frac{1}{2}, 1], \exists \in M_a(S) \).

As an application of Theorem 3.2, we obtain the following theorem which relates local quasi-invariance to absolute continuity.

3.7 Theorem. Let \( m \) be a left locally quasi-invariant positive regular Borel measure on \( S \), then \( L(S, m) \subseteq M_a'(S, S) \). (Here \( m \) is not necessarily bounded.)

Proof. Let \( \nu \in L(S, m) \) be fixed. We start by observing that if \( E \subseteq S \) is such that \( d(E) = E \) and \( E \) is compact, then

\[ d(E^n) = E^n \quad (n \in \omega). \]

It is sufficient to prove (1) for the case where \( n = 2 \), and an easy induction type of argument will yield the general case. Recalling that \( M_a(S) \) is norm closed in \( M(S) \) and \( \mu \ll \eta \) with \( \eta \in M_a(S) \) implies \( \mu \in M_a(S) \) (see \([2, 3.2 \text{ and } 2.4]\)), it is evident that if \( B \subseteq S \), then \( d(B) \) is closed and every point of \( d(B) \) is a cluster point of \( B \). In particular since \( E^2 \) is compact we then get \( d(E^2) \subseteq E^2 \). Now if \( x, y \in E \) and \( V \) is any Borel neighbourhood of \( xy \), there is a compact neighbourhood \( X \) of \( x \) with \( xy \subseteq Xy \subseteq V \). Since \( x \in d(E) \), there is positive measure \( \mu \in M_a(S) \) with \( \mu(X \cap E) > 0 \). Now \( \mu \cdot \tilde{y} \in M_a(S) \) \([2, 2.4 \text{ and } 3.6]\), and

\[ \mu \cdot \tilde{y}(V \cap E^2) \geq \mu \cdot \tilde{y}(Xy \cap E \cap y) \geq \mu(Xy^{-1} \cap E \cap y^{-1}) \geq \mu(X \cap E) > 0. \]

Hence \( xy \in d(E^2) \). So the equality \( d(E^2) = E^2 \) follows.

We now construct \( T \) such that \( T \subseteq d(T), \nu \in M(T), d(L_m) \cap T \neq \emptyset, 1 \) is an interior point of \( T \) and \( T \) is \( \sigma \)-compact.

By the regularity of \( \nu \) and the local compactness of \( S \), we can choose a sequence of compact sets \( \{C_n \} \subseteq S \) such that each \( C_n \) is the closure of an
open set (and so $d(C_n) = C_n$). $C_n \subset C_{n+1}$ and $\|\nu - \nu|_{C_n}\| < 1/n$ ($n \in \omega$).
Thus $\|\nu\| = \|\nu\|_{(\bigcup_{n=1}^{\infty} C_n)}$. Let $D$ be a compact set which is the closure of an open neighbourhood of the identity element $1$. Since $d(L_m) \neq \emptyset$ and clearly $d(L_m \setminus d(L_m)) = \emptyset$, there is a compact set $F \subset L_m$ such that $d(F) \neq \emptyset$.
Taking $K_n = C_n \cup D \cup d(F)$, we observe that $K_n$ is compact and $d(K_n) = K_n$ ($n \in \omega$). So by (1), $d((K_n)^k) = (K_n)^k$ ($n, k \in \omega$). Taking
$$T = \bigcup_{1 \leq k < \infty, 1 \leq n < \infty} (K_n)^k,$$
it is evident that $T$ is as stated above when we identify $\nu$ with $\nu|_{\bigcup_{n=1}^{\infty} C_n}$.

Since $M_a(S)_{\mathfrak{F}} \subseteq M_a(T)$ and $T \subseteq d(T)$, $T$ is thus a foundation semigroup with identity element $1$. Let $T_1$ be defined with respect to $T$ in a way $S_1$ was defined with respect to $S$ and observe that $T_1 \subseteq S_1$. We define the measure $m_T$ on $T$ by
$$m_T = \sum_{1 \leq k < \infty, 1 \leq n < \infty} 2^{-(n+k)} \frac{m|(K_n)^k}{1 + m((K_n)^k)}.$$Then $m_T \in M(T)$. Suppose $m_T(B) = 0$ for some Borel $B \subset T$. Then $m(B \cap T) = 0$ (of course Borel subsets of $T$ are also Borel subsets of $S$ since $T$ is).
Since $d(F) \subset L_m \cap T$, we then get
$$m(x^{-1}B \cap T) = m(x^{-1}(B \cap T)) = 0 \quad (x \in d(F)).$$So $m|(K_n)^k(x^{-1}B) = 0$ ($n, k \in \omega$ and $x \in d(F)$). Hence $m_T(x^{-1}B) = 0$ ($x \in d(F)$). If we define $d'$ with respect to $T$ as we defined $d$ with respect to $S$, we get $d'(L_m) \supseteq d(F)$. So $d'(L_m) \neq \emptyset$. By Proposition 2.2(i), we then have
$$\tilde{x} \ast m_T \in M_a(T) \quad (x \in T \setminus (d'(L_m) \cap L_m)).$$Clearly $\tilde{x} \ast \nu \ll \tilde{x} \ast m_T$ ($x \in T$), where $\nu$ is identified with $\nu|_{T}$. Consequently $\tilde{x} \ast \nu \in M_a(T)$ ($x \in T \setminus (d'(L_m) \cap L_m)$). Since $T_1$ is an ideal of $T$ and $T_1 \subseteq S_1$, we have proved that $\tilde{x} \ast \nu \in M_a(T)$ for some $x \in T \cap S_1$.
Thus $\nu \in M_a(T', T \cap S_1)$. By Theorem 3.2, it follows that $\nu \in M_a'(S, S_1)$. This completes our proof.

4. An example. We construct a commutative $S$ and a Borel measure $m$ on $S$ such that $L_m = S$ and supp($m$) = $S$, but $M_a(S) = \{0\}$—the zero subspace of $M(S)$.
Let $S = \{0\} \cup \bigcup_{n=0}^{\infty} \{n + 2^{-k} : k \in \omega\}$ with the relative topology of the line and multiplication given by $xy = \max(x, y)$ for all $x, y \in S$.

Let $\{x_n\}$ be an enumeration of $S$ and take $m = \Sigma_{n=1}^{\infty} 2^{-n}\delta_{x_n}$. Then $L_m = S$ and supp($m$) = $S$. However $M_a(S) = \{0\}$. To verify the last assertion the reader can check that if $x = n + 2^{-k}$, then the mapping $y \mapsto y \ast x(\{n + 2\})$
of $S$ into $\mathbb{R}$ is not continuous at $n + 2$ ($n, k \in \omega$).

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DEPARTMENT OF MATHEMATICS, THE UNIVERSITY, ABERDEEN, SCOTLAND

Current address: Mathematisch Instituut der Katholieke Universiteit, Toernooiveld, Nijmegen, The Netherlands