PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 69, Number 1, April 1978

ABSOLUTELY CONTINUOUS AND LOCALLY QUASI-INVARIANT MEASURES ON LOCALLY COMPACT SEMIGROUPS

HENRY A. M. DZINOTYIWEYI

ABSTRACT. Let S be a locally compact semigroup and T a locally compact subsemigroup of S. It is shown that, for a large class of semigroups S, if T is 'reasonably large', the absolutely continuous measures on T extend to measures whose translates by some points of S are absolutely continuous on S. As a consequence we prove the absolute continuity of translates of some measures related to a locally quasi-invariant measure.

1. Introduction. Throughout this paper, let S denote a locally compact jointly continuous topological semigroup, $M(S)$ the set of all bounded complex-valued Radon measures on S and $C_0(S)$ the set of all continuous functions on S which vanish at infinity. Identifying $M(S)$ as the first dual of $C_0(S)$, it is well known that $M(S)$ is a Banach algebra under the usual norm, $\| \; \|$, and convolution multiplication given by

$$\mu * \nu(f) = \int\int f(xy)\, d\mu(x)\, d\nu(x) \qquad (\mu, \nu \in M(S); f \in C_0(S)).$$

In the papers [1], [2], [3] and [7] certain subspaces of the algebra $M(S)$ are studied–the purpose being to find an analogue of $L^1(G)$ for a locally compact group G. Let $M_a^l(S)$ ($M_a^r(S)$) be the set of all measures μ in $M(S)$ such that the mapping $x \to \bar{x} * |\mu|$ ($x \to |\mu| * \bar{x}$) of S into $M(S)$ is weakly continuous, where $\bar{x}$ denotes the point mass at x and $|\mu|$ the total variation of μ. Then taking $M_a(S)$ to be $M_a^l(S) \cap M_a^r(S)$ we have the set studied in [2], [3] and [9]. In particular, from [2, Lemma 2.4 and Theorem 3.2] and [3, Theorem 2.6] we note that $M_a(S)$ is an *L-ideal* of $M(S)$ (i.e. $M_a(S)$ is a norm closed linear subspace of $M(S)$ such that for any $\nu, \mu \in M(S)$ we have (i) if $\nu \ll |\mu|$ and $\mu \in M_a(S)$ then $\nu \in M_a(S)$ and (ii) if $\nu \in M_a(S)$ then $\nu * \mu$, $\mu * \nu \in M_a(S)$). Further, when S is a group then $M_a(S)$ (or $M_a^l(S)$) can be identified with the usual group algebra $L^1(S)$ (see e.g. [6, 19.27 and 20.31]). A different approach is made in [7], where it is shown that if m is a positive measure on S such that $m(x^{-1}E) = m(Ex^{-1}) = 0$ whenever $m(E) = 0$ for all Borel $E \subset S$ and $x \in S$, then taking $L(S, m) = \{\nu \in M(S): \nu \ll m\}$, $L(S, m)$ has some

Received by the editors April 25, 1977 and, in revised form, August 11, 1977.
AMS (MOS) subject classifications (1970). Primary 43A05, 43A20; Secondary 43A35, 43A60.
Key words and phrases. Absolutely continuous measure, locally quasi-invariant measure, translate of a measure, foundation semigroup.

properties related to those of the usual group algebra $L^1(G)$ of a locally compact group G. It is natural to ask whether there is any relationship between $L(S, m)$ and $M_a(S)$. We will give a partial answer to this problem in Theorem 3.7. In §4 we will give an example of a semigroup S equal to the support of a measure m with the property (studied in [7] and) mentioned above but with $M_a(S)$ equal to zero.

Further, it is of interest to know the relationship between $M_a(T)$ and $M_a(S)$, when T is a locally compact subsemigroup of S. For example for those subsemigroups T of some semigroups S for which we are able to show that $M_a(T) = \{\nu \in M_a(S): |\nu|(S \setminus T) = 0\}$ (cf. Corollary 3.4) we have for instance: If S is commutative and equal to the carrier space of $M_a(S)$, then the bounded multiplicative linear functionals on $\{\nu \in M_a(S): |\nu|(S \setminus T) = 0\}$ are in one-to-one correspondence with the set of all continuous semicharacters on T. This follows easily from [3, Theorem 4.4] and one can easily note that [8, Theorem 13] is a special case of this result.

However, when S is not cancellative such subsemigroups T for which $M_a(T) = \{\nu \in M_a(S): |\nu|(S \setminus T) = 0\}$ are not easy to come by (see e.g. Remark 3.6). In that case we obtain a generalization of such an identification in Theorem 3.2.

We begin by collecting together some results and introducing further notation, in §2, which will be needed in §§3 and 4.

2. Preliminaries. For any sets $A, B \subseteq S$ and $x \in S$, let $AB = \{xy: x \in A, y \in B\}$, $A^{-1}B = \{y \in S: ay \in B$ for some $a \in A\}$, $x^{-1}B = \{x\}^{-1}B$, $A^{-1}x = A^{-1}\{x\}$ and similarly define AB^{-1}, Bx^{-1} and xA^{-1}. We say a measure μ is *absolutely continuous* if $\mu \in M_a(S)$. The measure $\bar{x} * \mu$ is the *left translate of* μ by $x \in S$ ($\mu \in M(S)$). For any regular Borel measure m on S, let $L_m = \{x \in S: |m|(B) = 0$ implies $m(x^{-1}B) = 0$ for all Borel $B \subset S\}$, $\mathrm{supp}(m) = \{x \in S: |m|(X) > 0$ whenever X is an open neighbourhood of $x\}$ and if $E \subset S$ is a Borel set we define the measure $m_{|E}$ by $m_{|E}(B) = m(B \cap E)$ for all Borel $B \subseteq S$. For $\mathcal{Q} \subseteq M(S)$ and Borel $E \subseteq S$ we let $\mathcal{Q}_{|E} = \{\mu \in \mathcal{Q}: |\mu|(S \setminus E) = 0\}$. For any $E \subseteq S$, let $d(E) = \{x \in S: x \in \mathrm{supp}(\mu_{|B})$ for some Borel set $B \subset E$ and $\mu \in M_a(S)\}$.

Following [2] and [3] we will say S is a *foundation semigroup* if $d(S) = S$, and following [4] we will say a regular Borel measure m is (*left*) *locally quasi-invariant* if $d(L_m) \neq \emptyset$.

Whenever S has an identity element 1, S_1 will denote the set of all $x \in S$ such that $X^{-1}x \cap xX^{-1}$ is a neighbourhood of 1, whenever X is a neighbourhood of x. This set S_1 plays a very significant role whenever S is a foundation semigroup with an identity element 1 (see e.g. [4] and [9]). In particular, from [9, 6.9 and 7.8] we have the following:

2.1 LEMMA. *If S is a foundation semigroup with an identity element 1, then S_1 is an ideal of S (i.e. $xy, yx \in S_1$ for all $x \in S_1$ and $y \in S$) and $d(S \setminus S_1) = 0$ (i.e. $M_a(S)$ is 'concentrated' on S_1).*

The next result was proved by us in [**4**].

2.2 PROPOSITION. *Let S be a foundation semigroup with identity element 1 and $\mu \in M(S)$. Then*

(i) $\bar{x} * \mu \in M_a(S)$, *for all* $x \in S_1(d(L_\mu) \cap L_\mu)$;

(ii) *if F is a subsemigroup containing a countable subset O_μ such that if $\bar{x} * |\mu|(B) = 0$ $(x \in O_\mu)$ then $\bar{x} * |\mu|(B) = 0$ $(x \in F)$, we have $\bar{x} * |\mu| \in M_a(S)$ $(x \in S_1(d(F) \cap F)F)$.*

We will need the next lemma in Theorem 3.2.

2.3 LEMMA. *If T is a locally compact subsemigroup of S, there is an isometric algebra isomorphism θ: $M(S)_{|T} \to M(T)$ given by $\theta(\mu)(B) = \mu(B \cap T)$ for all Borel $B \subseteq T$ and $\mu \in M(S)_{|T}$.*

PROOF. Since T is a locally compact subset of S, T is the intersection of an open subset and a closed subset of S (see e.g. [**11**, 18.4]). So T is a Borel subset of S. It is not hard to observe that the proof of [**12**, Theorem 3.4] contains the proof of our lemma. □

Finally for any $E \subset S$, let $M_a^l(S, E) = \{\mu \in M(S): \bar{x} * |\mu| \in M_a(S)$ for some $x \in E\}$ and note that $M_a^l(S, E)$ is a subalgebra of $M(S)$ such that if $\nu \in M(S)$, $\mu \in M_a^l(S, E)$ and $\nu \ll |\mu|$ then $\nu \in M_a^l(S, E)$. (This follows from the fact that $M_a(S)$ is an L-ideal as pointed out in the introduction.) Further if S is a group and $E \neq \varnothing$, then evidently $M_a^l(S, E) = M_a(S)$.

3. Main results. Throughout this section S is a foundation semigroup with an identity element 1 and T is a locally compact subsemigroup of S. We will only give proofs for the left-handed case, from which the right-handed and two-sided cases should be clear.

3.1 LEMMA. *Let E be a subset of S with $d(E)$ nonempty and $\mu \in M(S)$. If $\{\bar{x} * |\mu|: x \in E^n\}$ is a relatively weak compact subset of $M(S)$ $(n \in \omega)$ and $F = \bigcup_{n=1}^{\infty} E^n$, then*

$$\bar{x} * |\mu| \in M_a(S) \qquad \big(x \in S_1(d(F) \cap F)F\big).$$

Here $E^{n+1} = EE^n$ $(n \in \omega)$ and $E^1 = E$.

PROOF. By [**5**, Théorème 2], the relative weak compactness of $\{\bar{x} * |\mu|: x \in E^n\}$ implies that $\{\bar{x} * |\mu|: x \in E^n\}$ satisfies item (b) of the theorem of [**10**]. From [**10**, pp. 139 and 140] it follows that there is a countable set $A_n \subset E^n$ such that $\bar{x} * |\mu|(B) = 0$ $(x \in A_n)$ implies $\bar{x} * |\mu|(B) = 0$ $(x \in E^n)$.

Let $O_\mu = \bigcup_{n=1}^{\infty} A_n$ and observe that $\bar{x} * |\mu|(B) = 0$ $(x \in O_\mu)$ implies $\bar{x} * |\mu|(B) = 0$ $(x \in F)$. By Proposition 2.2(ii) we are done.

3.2 THEOREM. *Suppose $d(T) \neq \varnothing$. Then $M_a^l(T, T \cap S_1)$ is isometrically algebra isomorphic to*

$$M_a^l(S, T \cap S_1)_{|T} = \{\nu \in M_a^l(S, T \cap S_1): |\nu|(S \setminus T) = 0\}.$$

PROOF. Let θ be as stated in Lemma 2.3. Let $\nu \in M_a^l(S, S_1 \cap T)_{|T}$. There is $x \in T \cap S_1$ with $\bar{x} * |\nu| \in M_a(S)$. By [2, 2.4 and 3.2], $(\bar{x} * |\nu|)_{|T} \in M_a(S)$. Consequently $\theta((\bar{x} * |\nu|)_{|T}) \in M_a(T)$. Since $x \in T$, by the definition of θ, we get

$$\theta\big((\bar{x} * |\nu|)_{|T}\big) = \theta\big(\bar{x} * |\nu|\big) = \bar{x} * \theta\big(|\nu|\big) = \bar{x} * |\theta(\nu)|.$$

Thus $\theta(\nu) \in M_a^l(T, T \cap S_1)$.

On the other hand suppose $\nu' \in M_a^l(T, T \cap S_1)$. There is a unique ν in $M(S)_{|T}$, such that $\theta(\nu) = \nu'$, by Lemma 2.3. There is $x_0 \in T \cap S_1$ such that $\bar{x}_0 * |\nu'| \in M_a(T)$. Since $d(T) \neq \varnothing$, there is a measure $\eta \in M_a(S)$ such that $\eta(T) > 0$. By the regularity of η, we can find a compact subset E contained in T, such that $\eta(E) > 0$. Thus $d(E) \neq \varnothing$. Observe that E^n is a compact subset of T ($n \in \omega$). So, by our definition of $M_a(T)$, $\{\bar{y} * (\bar{x}_0 * |\nu'|): y \in E^n\}$ is a weakly compact subset of $M(T)$ ($n \in \omega$). For all $x, y \in T$ we have

$$\bar{y} * \big(\bar{x} * |\nu'|\big) = \bar{y} * \bar{x} * \theta\big(|\nu|\big) = \theta\big(\bar{y} * (\bar{x} * |\nu|)\big).$$

It follows that $\{(\bar{y} * (\bar{x}_0 * |\nu|))_{|T}: y \in E^n\}$ is a weakly compact subset of $M(T)$ and hence of $M(S)$, by Hahn-Banach Theorem. But since $|\nu|(S \setminus T) = 0$, we have

$$\begin{aligned}
\big(\bar{y} * (\bar{x}_0 * |\nu|)\big)_{|T} &= \big(\overline{yx}_0 * |\nu|\big)_{|T}(B) \\
&= \big(\overline{yx}_0 * |\nu|\big)(B \cap T) + |\nu|(S \setminus T) \\
&= \overline{yx}_0 * |\nu|(B \cap T) + \overline{yx}_0 * |\nu|(S \setminus T) \\
&= \overline{yx}_0 * |\nu|(B) \\
&= \bar{y} * \big(\bar{x}_0 * |\nu|\big)(B),
\end{aligned}$$

for every Borel $B \subset S$ and $y \in E^n$. It follows that $\{\bar{y} * (\bar{x}_0 * |\nu|): y \in E^n\}$ is a weakly compact subset of $M(S)$ ($n \in \omega$). By Lemma 3.1 we then get

$$\bar{x} * |\nu| \in M_a(S) \qquad \big(x \in S_1(d(F) \cap F)Fx_0\big),$$

where $F = \bigcup_{n=1}^{\infty} E^n$. Since $d(S \setminus S_1) = \varnothing$ and $d(T) \neq \varnothing$, we have $S_1 \cap T \neq \varnothing$. Since T is a subsemigroup and S_1 is an ideal of S,

$$(S_1 \cap T)(d(F) \cap F)Fx_0 \subseteq T \cap S_1.$$

Consequently $\nu \in M_a^l(S, T \cap S_1)$.

It is now evident from Lemma 2.3 that the restriction of θ to $M_a^l(S, T \cap S_1)_{|T}$ is the required isometric algebra isomorphism.

3.3 Corollary. *If $d(T) \neq \varnothing$, then $M_a(T) \subseteq M_a^l(S, S_1)$.*

3.4 Corollary. *Suppose S is concellative (i.e. $xy = xz$ or $yx = zx$ implies $y = z$ ($x, y, z \in S$)) and $d(T) \neq \varnothing$. Then*

$$M_a(T) = \{\nu \in M_a(S): |\nu|(S \setminus T) = 0\}.$$

Proof. Evidently $M_a(S)_{|T} \subseteq M_a(T)$ so we only need to show that $M_a(T) \subseteq M_a(S)_{|T}$. Let $\nu \in M_a(T)$, K a compact subset of S and suppose $x_\alpha \to x$ in S. By Theorem 3.2, there is $x_0 \in S_1$ with $\bar{x}_0 * |\nu| \in M_a(S)$. Hence

$$\big||\nu|(Kx_\alpha^{-1}) - |\nu|(Kx^{-1})\big| = \big||\nu|(x_0^{-1}(x_0K)x_\alpha^{-1}) - |\nu|(x_0^{-1}(x_0K))x^{-1}\big|$$

by cancellation

$$= \big|\bar{x}_0 * |\nu|((x_0K)x_\alpha^{-1}) - \bar{x}_0 * |\nu|((x_0K)x^{-1})\big| \to 0.$$

This implies that $\nu \in M_a^r(S)$, by [2, Lemma 2.4]. Similarly $\nu \in M_a^l(S)$ and hence $\nu \in M_a(S)_{|T}$.

In particular, we have the following improvement of [1, Proposition 1.16(iii)]:

3.5 Corollary. *Let S be a group with a Haar measure m. If $m(T) \neq 0$, then $M_a(T) = \{\nu \in M(S)\colon\ \nu \ll m \text{ and } |\nu|(S \setminus T) = 0\}$ (cf. [9, Theorem 4.11]).*

Proof. This follows from Corollary 3.4 and the fact that $M_a(S) = L(S, m)$ (see [6, 19.27 and 20.31]).

3.6 Remark. In general no identification of $M_a(T)$ and $M_a(S)_{|T}$ can be achieved even when S is compact commutative and $d(T) = T$.

For example consider $S = [0, 1]$ with the usual topology and the operation $xy = \min(x + y, 1)$. This example is borrowed from [9]. S is evidently a foundation semigroup. Take $T = [\frac{1}{2}, 1]$ and observe that $M_a(T) = M(T)$. However, for all $x \in [\frac{1}{2}, 1)$, $\bar{x} \notin M_a(S)$.

As an application of Theorem 3.2, we obtain the following theorem which relates local quasi-invariance to absolute continuity.

3.7 Theorem. *Let m be a left locally quasi-invariant positive regular Borel measure on S, then $L(S, m) \subseteq M_a^l(S, S_1)$. (Here m is not necessarily bounded.)*

Proof. Let $\nu \in L(S, m)$ be fixed. We start by observing that if $E \subset S$ is such that $d(E) = E$ and E is compact, then

$$d(E^n) = E^n \qquad (n \in \omega). \tag{1}$$

It is sufficient to prove (1) for the case where $n = 2$, and an easy induction type of argument will yield the general case. Recalling that $M_a(S)$ is norm closed in $M(S)$ and $\mu \ll \eta$ with $\eta \in M_a(S)$ implies $\mu \in M_a(S)$ (see [2, 3.2 and 2.4]), it is evident that if $B \subset S$, then $d(B)$ is closed and every point of $d(B)$ is a cluster point of B. In particular since E^2 is compact we then get $d(E^2) \subseteq E^2$. Now if $x, y \in E$ and V is any Borel neighbourhood of xy, there is a compact neighbourhood X of x with $xy \in Xy \subset V$. Since $x \in d(E)$, there is positive measure $\mu \in M_a(S)$ with $\mu(X \cap E) > 0$. Now $\mu * \bar{y} \in M_a(S)$ [2, 2.4 and 3.6], and

$$\mu * \bar{y}(V \cap E^2) \geqslant \mu * \bar{y}(Xy \cap Ey) \geqslant \mu(Xyy^{-1} \cap Eyy^{-1}) \geqslant \mu(X \cap E) > 0.$$

Hence $xy \in d(E^2)$. So the equality $d(E^2) = E^2$ follows.

We now construct T such that $T \subseteq d(T)$, $\nu \in M(T)$, $d(L_m) \cap T \neq \varnothing$, 1 is an interior point of T and T is σ-compact.

By the regularity of ν and the local compactness of S, we can choose a sequence of compact sets $\{C_n\} \subset S$ such that each C_n is the closure of an open set (and so $d(C_n) = C_n$), $C_n \subset C_{n+1}$ and $\|\nu - \nu_{|C_n}\| < 1/n$ $(n \in \omega)$. Thus $\|\nu\| = |\nu|(\bigcup_{n=1}^{\infty} C_n)$. Let D be a compact set which is the closure of an open neighbourhood of the identity element 1. Since $d(L_m) \neq \emptyset$ and clearly $d(L_m \setminus d(L_m)) = \emptyset$, there is a compact set $F \subset L_m$ such that $d(F) \neq \emptyset$. Taking $K_n = C_n \cup D \cup d(F)$, we observe that K_n is compact and $d(K_n) = K_n$ $(n \in \omega)$. So by (1), $d((K_n)^k) = (K_n)^k$ $(n, k \in \omega)$. Taking

$$T = \bigcup_{\substack{1 \leqslant k < \infty \\ 1 \leqslant n < \infty}} (K_n)^k,$$

it is evident that T is as stated above when we identify ν with $\nu_{|\bigcup_{n=1}^{\infty} C_n}$.

Since $M_a(S)_{|T} \subseteq M_a(T)$ and $T \subseteq d(T)$, T is thus a foundation semigroup with identity element 1. Let T_1 be defined with respect to T in a way S_1 was defined with respect to S and observe that $T_1 \subseteq S_1$. We define the measure m_T on T by

$$m_T = \sum_{\substack{1 \leqslant k < \infty \\ 1 \leqslant n < \infty}} 2^{-(n+k)} \frac{m_{|(K_n)^k}}{1 + m\big((K_n)^k\big)}.$$

Then $m_T \in M(T)$. Suppose $m_T(B) = 0$ for some Borel $B \subset T$. Then $m(B \cap T) = 0$ (of course Borel subsets of T are also Borel subsets of S since T is). Since $d(F) \subset L_m \cap T$, we then get

$$m(x^{-1}B \cap T) \leqslant m\big(x^{-1}(B \cap T)\big) = 0 \qquad (x \in d(F)).$$

So $m_{|(K_n)^k}(x^{-1}B) = 0$ $(n, k \in \omega$ and $x \in d(F))$. Hence $m_T(x^{-1}B) = 0$ $(x \in d(F))$. If we define d' with respect to T as we defined d with respect to S, we get $d'(L_{m_T}) \supseteq d(F)$. So $d'(L_{m_T}) \neq \emptyset$. By Proposition 2.2(i), we then have

$$\bar{x} * m_T \in M_a(T) \qquad \big(x \in T_1\big(d'(L_{m_T}) \cap L_{m_T}\big)\big).$$

Clearly $\bar{x} * |\nu| \ll \bar{x} * m_T$ $(x \in T)$, where ν is identified with $\nu_{|T}$. Consequently $\bar{x} * |\nu| \in M_a(T)$ $(x \in T_1(d'(L_{m_T}) \cap L_{m_T}))$. Since T_1 is an ideal of T and $T_1 \subset S_1$, we have proved that $\bar{x} * |\nu| \in M_a(T)$ for some $x \in T \cap S_1$. Thus $\nu \in M_a^l(T, T \cap S_1)$. By Theorem 3.2, it follows that $\nu \in M_a^l(S, S_1)$. This completes our proof.

4. An example. We construct a commutative S and a Borel measure m on S such that $L_m = S$ and $\mathrm{supp}(m) = S$, but $M_a(S) = \{0\}$–the zero subspace of $M(S)$.

Let $S = \{0\} \cup \bigcup_{n=0}^{\infty}(\{n + 2^{-k}\colon k \in \omega\})$ with the relative topology of the line and multiplication given by $xy = \max(x, y)$ for all $x, y \in S$.

S ....

0   1   2   3   4   etc.

Let $\{x_n\}$ be an enumeration of S and take $m = \sum_{n=1}^{\infty} 2^{-n}\bar{x}_n$. Then $L_m = S$ and $\mathrm{supp}(m) = S$. However $M_a(S) = \{0\}$. To verify the last assertion the reader can check that if $x = n + 2^{-k}$, then the mapping $y \to \bar{y} * \bar{x}(\{n + 2\})$

of S into $\mathbf{R}$ is not continuous at $n + 2$ $(n, k \in \omega)$.

I am indebted to Dr. A. L. T. Paterson for useful discussions.

References

1. A. C. Baker and J. W. Baker, *Algebras of measures on a locally compact semigroup*, J. London Math. Soc. **1** (1969), 249–259. MR **39** #7019.

2. ______, *Algebras of measures on a locally compact semigroup*. II, J. London Math. Soc. **2** (1970), 651–659. MR **43** #402.

3. ______, *Algebras of measures on a locally compact semigroup*. III, J. London Math. Soc. **4** (1972), 685–695. MR **46** #5928.

4. H. A. M. Dzinotyiweyi, *Locally quasi-invariant measures*, J. London Math. Soc. **15** (1977), 484–488.

5. A. Grothendieck, *Sur les applications linéaires faiblement compactes d'espaces du type* $C(K)$, Canad. J. Math. **5** (1953), 129–173. MR **15**, 438.

6. E. Hewitt and K. A. Ross, *Abstract harmonic analysis*. I, Springer-Verlag, Berlin, 1963. MR **28** #158.

7. N. J. Rothman, *An* L^1*-algebra for certain locally compact topological semigroups*, Pacific J. Math. **23** (1967), 143–151. MR **35** #5955.

8. N. J. Rothman and M. W. Schuh, *Laplace transforms on vanishing algebras*, Semigroup Forum **8** (1974), 189–214. MR **51** #5853.

9. G. L. G. Sleijpen, *Convolution measure algebras on semigroups*, Thesis, Katholieke Universiteit, Toernooiveld, Nijmegen, 1976.

10. J. D. Stein, *Uniform absolute continuity in spaces of set functions*, Proc. Amer. Math. Soc. **51** (1975), 137–140.

11. S. Willard, *General topology*, Addison-Wesley, Reading, Mass. 1970. MR **41** #9173.

12. J. C. S. Wong, *Absolutely continuous measures on locally compact semigroups*, Canad. Math. Bull. **18** (1975), 127–132. MR **52** #1168.

Department of Mathematics, The University, Aberdeen, Scotland

Current address: Mathematisch Instituut der Katholieke Universiteit, Toernooiveld, Nijmegen, The Netherlands