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NICE SETS OF MULTI-INDICES

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ABSTRACT. Finite sets, A, of n-tuples for which $(\Sigma_{\alpha\in A}(\Pi_{j=1}^n|x_j|^{\alpha_j}))^{-p}, p > 0$, is integrable over R^n are given a simple characterization. Applications to certain Fourier multiplier theorems are mentioned.

Let A be a finite subset of R^n and consider the function h_A, defined on R^n by the formula $h_A(x) = \Sigma_{\alpha\in A}(\Pi_{i=1}^n|x_i|^{\alpha_i})$. It is sometimes useful to know when h_A^{-p}, $p > 0$, is integrable over R^n. When this is the case we call A p-nice.

For example consider those finite subsets A of R^n whose elements, α, have components which are nonnegative integers and associate with each such α the derivative of order α in the usual manner. Given a distribution f on R^n, the integrability of h_A^{-2} determines the derivatives of f in $L^2(R^n)$ needed to conclude that f is a continuous function as in a classical theorem of Sobolev. The integrability of h_A^{-2} also determines the derivatives one needs to compute in order to apply certain variants of the Fourier multiplier theorem of Marcinkiewicz (see [2]).

Let v denote that element of R^n all of whose components are one.

PROPOSITION. *A finite subset A of R^n is p-nice if and only if v/p is contained in the interior of the convex hull of A.*

To see that the condition on v/p is sufficient let e_j, $j = 1, \ldots, 2^n$, denote those elements of R^n whose components are either one or minus one. By hypothesis there is a positive ε such that $(v/p) + \varepsilon e_j = \Sigma_{\alpha\in A} r_{j,\alpha}\alpha$, where $r_{j,\alpha} \geqslant 0$ and $\Sigma_{\alpha\in A} r_{j,\alpha} = 1, j = 1, \ldots, 2^n$. Since $h_A(x) \geqslant \Sigma_{\alpha\in A} r_{j,\alpha}(\Pi|x_j|^{\alpha_j})$, an application of the inequality between the arithmetic and geometric mean results in the 2^n inequalities

$$(1) \qquad (h_A(x))^{-p} \leqslant \prod_{j=1}^{n} \left(|x_j|^{-1\pm p\varepsilon}\right).$$

It is clear from the above inequalities that $(h_A(x))^{-p}$ is integrable over R^n.

Suppose v/p is not contained in the interior of the convex hull of A. Then there is a hyperplane which separates v/p and the convex hull of A. More specifically, there is a u in R^n such that $\alpha \cdot u \leqslant (v \cdot u)/p$ for all $\alpha \in A$.

There is no loss of generality if we assume, which we do, that u and all the

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elements of A have nonnegative components. For if T is the transformation which maps $x = (x_1, \ldots, x_n)$ into $(\gamma_1 x_1, \ldots, \gamma_n x_n)$, $\gamma_i \neq 0$, $i = 1, \ldots, n$, and $B = \{T(\alpha - v/p) + v/p: \alpha \in A\}$, then $\beta \cdot T^{-1}u \leqslant (v \cdot T^{-1}u)/p$ for all β in B and

$$\int_{R^n} (h_B(x))^{-p}\, dx = \left|\prod_{i=1}^{n} \gamma_i\right|^{-1} \int_{R^n} (h_A(x))^{-p}\, dx.$$

Furthermore, it is clear that if u and A do not satisfy the desired property then $\gamma_1, \ldots, \gamma_n$ can be chosen so that $T^{-1}u$ and B do.

If none of the components of u are zero, then A is contained in the convex hull of the origin and the points $\beta_1, \ldots, \beta_n$, where the jth component of β_j is $b_j = u \cdot v/pu_j$ and the remaining components are zero, $j = 1, \ldots, n$. Again an application of the inequality between the arithmetic and geometric mean results in $\prod_{j=1}^n |x_j|^{\alpha_j} \leqslant 1 + \sum_{j=1}^n |x_j|^{b_j}$, for all $\alpha = (\alpha_1, \ldots, \alpha_n)$ in A. The last inequality, together with a polar change of variables or direct calculation, implies that

$$\int_{R^n} (h_A(x))^{-p}\, dx \geqslant C \int_0^\infty (1 + r)^p r^{\lambda - 1}\, dr$$

where $\lambda = \sum_{j=1}^n b_j^{-1}$ and C is a positive constant which depends on A and p. Since $\sum_{j=1}^n b_j^{-1} = p$, the last integral diverges.

If one of the components of u, say u_1, is zero, proceed by induction. Namely, given x in R^n denote by x' that element of R^{n-1} obtained by deleting the first component of x and let $A' = \{\alpha': \alpha \in A\}$. Observe that $\int R^n (h_A(x))^{-p}\, dx \geqslant \int_{R^{n-1}} (h_{A'}(x'))^{-p}\, dx'$. Since v'/p is not in the interior of the convex hull of A', the last integral diverges. This completes the proof.

Some examples of nice sets are given in [5]. An application of the proposition shows that the set $A = \{\alpha = (\alpha_1, \ldots, \alpha_n)$: $\alpha_i = 0$ or 1 and $\sum \alpha_i \leqslant \kappa$, where κ is the least integer greater than $n/p\}$ is p-nice. (Note that the usage of the term nice in [5], where only sets of multi-indices are considered, is somewhat different from ours. In fact, if a set of multi-indices is p-nice and $\alpha \in A$ implies that any multi-index β satisfying $\beta \leqslant \alpha$ is also in A, we call A very p-nice.)

As another application, consider the multiplier theorem of Hormander [2, p. 120] and let A denote the set of multi-indices in the hypothesis of that theorem. Recalling the proof, besides the fact that A is very 2-nice, the only other property of A which is used is that $\int_{|x|>t} [h_A(x)]^{-2}\, dx \leqslant Ct^{-\delta}$, for some positive constants C and δ and all positive t. A similar inequality holds for any p-nice set. In fact, more generally, if S is a linear transformation on R^n satisfying $Sx \cdot x \geqslant |x|^2$ and $T_t = \exp(S \log t)$, $t > 0$, is the group of linear transformations with infinitesimal generator S, then we have the following

COROLLARY. *If A is a p-nice set then there are positive constants C and δ such that* $\int_{B_t} (h_A(x))^{-p}\, dx \leqslant Ct^{-\delta}$, $t > 0$, *where* $B_t = \{x \in R^n: |T_{t^{-1}} x| \geqslant 1\}$.

If Q_t denotes the complement of the cube $\{x: |x_i| \leqslant t/\sqrt{n}\}$ then $Q_t \supset B_t$,

$t > 1$, and hence it suffices to verify the inequality with B_t replaced by Q_t. The 2^n inequalities labeled (1) in the proof of the proposition imply that $h_A(x)^{-p} \leqslant C_1 \prod_{j=1}^n h(x_j)$, where $h(s) = (1 + |s|)^{-2\delta}|s|^{-1+\delta}$, for $-\infty < s < \infty$. A direct calculation shows that

$$\int_{Q_t} (h_A(x))^{-p}\, dx \leqslant C_1[2^n - 1]\left(\int_{-\infty}^{\infty} h(s)\, ds\right)^{n-1} \int_{t/\sqrt{n}}^{\infty} s^{-1-\delta}\, ds = Ct^{-\delta}.$$

It follows from the corollary that Hormander's multiplier theorem and its variants are true when the conditions on the set of multi-indices is somewhat relaxed. (See **[1]**, **[3]**, **[4]**, **[6]**, **[7]**.) As a specific example we mention the following

THEOREM. *Suppose $m \in L^\infty(R^n)$, A is a very q-nice set of multi-indices, and $\sum_{\alpha \in A} \int_{1<|\xi|<2} |D^\alpha m_t(\xi)|^q\, d\xi \leqslant B^q$ for all $t > 0$, where $m_t(\xi) = m(T_t\xi)$ and q is a number such that $1 \leqslant q \leqslant 2$. Then m is a Fourier multiplier from $L^p(R^n)$ to $L^p(R^n)$ with multiplier norm bounded by $CBp^2(p-1)^{-1}$ for all p, $1 < p < \infty$, where C is a constant depending only on n. Furthermore, if H^1 is the parabolic Hardy space defined with respect to the group of "dilations" $T_t^* = \exp(S^* \log t)$, where S^* is the transpose of S, as in* **[1]**, *then m is a Fourier multiplier from H^1 to H^1 with multiplier norm bounded by CB, where C depends only on the choice of norm in H^1.*

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