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BOUNDED SECTIONS ON A RIEMANN SURFACE

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ABSTRACT. Let X denote a hyperbolic Riemann surface, ζ a unitary line bundle, and $H^{\infty}(\zeta)$ the Banach space of bounded holomorphic sections of ζ. If, for a given point ξ in X, the norm of the evaluation functional on $H^{\infty}(\zeta)$ varies continuously with the bundle ζ, then it is shown that the space of bounded holomorphic sections is dense in the space of holomorphic sections for every unitary line bundle.

The results of this note are related to the work of Harold Widom in [1] and [2]. They center around the following data–a connected hyperbolic Riemann surface X, and equivalence class of unitary line bundles ζ on X, and the space $B(X, \zeta)$ of bounded holomorphic sections of ζ. The function m_{∞} is defined as $m_{\infty}(X, \zeta, z) = \sup|f(z)|$ where $f \in B(X, \zeta)$ and $|f| \leqslant 1$. The function m_{∞} is a map from G^*, the group of equivalence classes of unitary line bundles, to the interval $0 \leqslant x \leqslant 1$. The group G^* is the character group of the first homology group $H_1(X, Z)$ and as such is given the weak topology. In [2] there are several interesting consequences of the assumption that m_{∞} be continuous. The surfaces for which this is true have properties similar to the disk as concerns bounded holomorphic functions. One of these properties is that such a surface X is B-convex, i.e., the space $B(X)$ of bounded holomorphic functions is dense in the space $H(X)$ of holomorphic functions. This is established below.

THEOREM. *If the map m_{∞} is continuous, then*
(a) *X is B-convex,*
(b) *$B(X)$ is dense in $H_p(X)$ for $p > 0$ and in $N^+(X)$, and*
(c) *X is regular for potential theory.*

The space $N^+(X)$ is the Smirnov class for X and $N(X)$ is the Nevanlinna class. It should be noted that in [2] it is asserted that if m_{∞} is continuous, then $B(X)$ is dense in $N(X)$. This cannot be true in general. The argument there proves the assertion that $B(X)$ is dense in $N^+(X)$. Even when the surface is a disk, B is not dense in N since N^+, which contains B, is a closed proper subspace of N (see [3] for the definitions and facts; especially p. 919).

The proof of (b) appears in [2]. The proof of (c) follows from (a) and recent work of Hasumi in the following way. Results of [4, p. 276] show that

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whenever every $B(X, \zeta)$ has nonzero members (which is the case when m_∞ is continuous), then X is a surface Y less a discrete set and Y is regular for potential theory. If (a) holds, then clearly $X = Y$.

Now I prove (a). A necessary condition for the continuity of m_∞ is that every unitary line bundle have nonzero bounded holomorphic sections. This in turn means that $B(X)$ separates points and provides local coordinates. To show the B-convexity of X I use the properties of algebras of holomorphic functions on a Riemann surface as developed in [5], [6] and [7]. The algebra to which these results will be applied is $l(X)$, the closure in $H(X)$ of the bounded holomorphic functions. Using [6, Theorem 2, p. 508] there is a surface Y and a morphism ϕ from X to Y and an algebra A' on Y and a map τ from $l(X)$ to A' which is bijective and $(\tau(f)) \circ \phi = f$. Because $l(X)$ separates, ϕ is injective and I may suppose $X \subseteq Y$, ϕ is inclusion and τ is restriction to X. Condition (i) of [6, Theorem 2, p. 508] implies that for every compact set $k \subset Y$ there is a compact set k_0 in X for which

$$k \subset \{z \mid |f(z)| \leqslant \|f\|_{k_0} \ \forall f \in A'\}. \tag{1}$$

If $f \in B(X)$, then f extends to a member of A'; hence on k, $|f(z)| \leqslant \|f\|_{k_0} \leqslant \|f\|_X$. Since k is an arbitrary compact set of $Y, f \in B(Y)$ and the norm of f on Y is the same as its norm on X. So $B(X)$ is $B(Y)$ restricted to X. Also I claim that A' is $l(Y)$. Since $B(X)$ and $B(Y)$ are identical $l(Y) \subseteq A'$ and A' is the set of $F \in H(Y)$, $F|_X \in l(X)$. So F is the uniform limit on compact subsets of X of members of $B(X)$. Condition (1) and the fact that $B(X) = B(Y)|_X$ show that convergence on X extends to Y. So $A' = l(Y)$. Using (iv) of [6, Theorem 2, p. 508] for every compact k,

$$k(A') = \{z \mid z \in Y, |f(z)| \leqslant \|f\|_k \ \forall f \in A'\} \tag{2}$$

is compact and consists of k union with those components of the complement of k which are relatively compact. Since $B(Y)$ closure is $l(Y) = A'$, $\hat{k}(A') = \hat{k}(B(Y))$.

Now I show that $X = Y$. In what follows the relation between sections of unitary line bundles and multifunctions which are holomorphic and whose absolute values are functions will be exploited as in [1]. Two multifunctions whose absolute values are functions give rise to the same section provided they have identical moduli. Let a belong to the closure of X but not to X and let $0 \leqslant t \leqslant 1$. There is a holomorphic multifunction f_t on $Y - \{a\}$ such that $|f_t| = \exp(-t \cdot g_a)$; and f_t corresponds to a section of some ξ_t on $Y - \{a\}$. I can restrict ξ_t to X and choose an $h \in B(X, \bar{\xi}_t)$ so $|h| \leqslant 1$ and h is not identically zero. Then $hf_t = F \in B(X)$ so extends to $B(Y)$. On Y, $|F| = \exp(-u - p)$, where u is a positive harmonic function and p is a discrete Green potential; because $|F| \leqslant 1$ on X means $|h| = \exp(-u - p + tg_a) \leqslant 1$ on $Y - \{a\}$ so $u + p - tg_a \geqslant 0$ on $Y - \{a\}$ or $u + p \geqslant tg_a$ and this last inequality holds on Y. This can be the case only if $p = g_a + q$ where q is a discrete Green potential. Therefore

$$|h| = \exp(-u - g_a - q + tg_a) \leqslant \exp(-(1 - t) g_a).$$

Hence,

(1) $$m_\infty(X, \bar{\xi}_t, z) \leqslant \exp(-(1-t)g_a(z)).$$

As $t \to 0$, $f_t \to 1$ so $\xi_t \to i_d$ and $\bar{\xi}_t \to i_d$. The assumed continuity of m_∞ on X means $m_\infty(X, \bar{\xi}_t, z)$ increases to 1. But from (1), $1 \leqslant \exp(-g_a(z)) < 1$, a contradiction. Thus $X = Y$.

Now it follows that for every compact $k \subset X$, $\hat{k}_\infty = \{z \mid |f(z)| \leqslant \|f\|_k \ \forall f \in B(X)\}$ is compact and equals $\hat{k} = \{z \mid |f(z)| \leqslant \|f\|_k \ \forall f \in H(X)\}$. Moreover, X can be exhausted by compact sets k for which $k = \hat{k} = \hat{k}_\infty$ [8, p. 240]. The result of Bishop [5, Corollary 2, p. 48] can be applied to conclude that on any such compact set $k = \hat{k} = \hat{k}_\infty$ every function continuous on k and holomorphic on its interior is the uniform limit of members of $I(X)$, hence of members of $B(X)$. Thus X is B-convex.

Now I can use the result that X is B-convex to prove the stronger statement that $B(X, \zeta)$ is dense in $H(X, \zeta)$ for every unitary line bundle ζ.

THEOREM. *If m_∞ is continuous, then $B(X, \zeta)$ is dense in $H(X, \zeta)$ for every ζ.*

The first step in the proof is to show that for every compact set K and every unitary line bundle ζ there is a section $f \in B(X, \zeta)$ which has no zeros on K. Let $A = \{\zeta \mid$ there is an $f \in B(X, \zeta)$ with no zeros on $K\}$. A is a subset of G^* and has these properties–$i_d \in A$; if $\xi, \eta \in A$, then $\xi\eta \in A$; and A contains an open set about i_d. Only the last property needs verification. I suppose that each neighborhood of i_d has a ξ with the property that every $f \in B(X, \xi)$ has a zero on K. In particular, if f is an extremal section at a for ξ normalized so $f(a) = m_\infty(X, \xi, a)$, then f has a zero on K for certain ξ close to i_d. As a consequence there is a sequence of unitary line bundles $\{\xi_j\}$ converging to i_d and a corresponding sequence of points $\{z_j\}$ in K such that the corresponding extremal f_j for ξ_j at a satisfies $|f_j(z_j)| = 0$. I may assume that the sequence $\{z_j\}$ converges to some $b \in K$ and at the same time, that $\{f_j\}$ converges on K uniformly to a function $f \in B(X)$. By the assumed continuity of m_∞, $m_\infty(\xi_j, a) = f_j(a)$ converges to 1. Since $|f| \leqslant 1$ and $f(a) = 1$, I must have $f \equiv 1$. Now b is a limit of $\{z_j\}$ so $|f(b)| = \lim|f_j(z_j)| = 0$ which is a contradiction. Since G^* is both compact and connected it follows that $G^* = A$.

To conclude the proof of the theorem let K denote a compact set such that $K = \hat{K}_\infty = \hat{K}$ (the conclusion of Theorem 1 and the fact that X is a Stein manifold means X can be exhausted by such sets). Let $f \in H(X, \xi)$ and let $b \in B(X, \xi)$ so b has no zeros on a neighborhood of K. Let $\varepsilon > 0$ be given. Since $K = \hat{K}$ and since X is holomorph-convex and f/b is holomorphic on a neighborhood of K, there is an $F \in H(X)$ for which $|F - f/b| < \varepsilon\|b\|^{-1}$ on K. As X is B-convex I may take it that $F \in B(X)$. Then $|f - bF| < \varepsilon$ on K. Since $bF \in B(X, \xi)$, the assertion of the theorem follows.

Referring to [9] one can remove from the punctured disk $0 < |z| < 1$ a sequence of closed disks so that the resulting surface X has the property that the origin is not a regular point for potential theory. On the other hand X is B-convex. This is easily seen to be the case since $B(X)$ separates, provides local coordinates, and $\hat{k}_\infty$ is compact for every compact k. It follows that X is B-convex but m_∞ is not continuous.

One can interpret m_∞ in terms of linear functionals in the following way. If F is a section of $\bar{\zeta}$ such that $|F(\xi)| = 1$, then the map which sends f onto $(Ff)(\xi) = L_\zeta(f)$ is a continuous linear functional on $H^\infty(X, \zeta)$ ($B(X, \zeta)$ with the norm topology). Then $m_\infty(\zeta) = \|L_\zeta\|$; so the continuity of m_∞ means that the norm varies continuously with ζ, the unitary line bundle.

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