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THE CORONA THEOREM AS AN OPERATOR THEOREM[1]

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ABSTRACT. We provide a short proof of a theorem of W. B. Arveson in operator theory. The conclusion of this theorem is the same as that of the Corona Theorem but the hypotheses are operator theoretic. Our proof yields an exact value for the constant involved. We also comment on this theorem as a new approximation problem.

If H^∞ denotes the bounded analytic functions on the unit disk D, then Carleson's Corona Theorem [2] states that given $f_1, \ldots, f_n \in H^\infty$ there exist $g_1, \ldots, g_n \in H^\infty$ such that $\Sigma_j f_j g_j = 1$ if and only if for some $\varepsilon > 0$, $\Sigma_j |f_j(z)|^2 \geqslant \varepsilon$ for all $z \in D$. Carleson also gives an upper bound for each of $|g_1|, \ldots, |g_n|$. In [1] W. B. Arveson has proved a theorem in operator theory whose conclusion is similar to that of the Corona Theorem but with different bounds on $|g_1|, \ldots, |g_n|$, since the hypotheses are operator theoretic. To state and prove this theorem we shall need the elementary facts about Toeplitz operators that we review below; see also [3, Chapters 6, 7].

Let L^2 and L^∞ be, respectively, the spaces of square integrable and essentially bounded functions on the unit circle with norms $\|\cdot\|$ and $\|\cdot\|_\infty$. If P denotes the orthogonal projection of L^2 onto the subspace of functions whose Fourier series contain only nonnegative powers of $e^{i\theta}$, then we may identify the usual Hardy space H^2 with PL^2 and identify H^∞ with $H^2 \cap L^\infty$. Every $\varphi \in L^\infty$ generates a bounded operator M_φ on L^2 by $M_\varphi u = \varphi u$ for all $u \in L^2$ and for this operator $\|M_\varphi\| = \|\varphi\|_\infty$. Every $\varphi \in H^\infty$ generates an analytic Toeplitz operator on H^2 by $T_\varphi u = \varphi u$ for all $u \in H^2$ and again $\|T_\varphi\| = \|\varphi\|_\infty$. When both φ and ψ are in H^∞ then $T_{\varphi\psi} = T_\varphi T_\psi = T_\psi T_\varphi$. For the particular case $\psi(z) = z$, denote T_z by S, then $T_\varphi S = S T_\varphi$ and S is an isometry, in fact a unilateral shift of multiplicity one on H^2. Finally if the conjugate of T_φ is T_φ^*, then $T_\varphi^* = PM_{\bar\varphi}|H^2$. With these preliminaries out of the way, the result that we prove is the following.

THEOREM. *If $f_1, \ldots, f_n \in H^\infty$, then there exist $g_1, \ldots, g_n \in H^\infty$ satisfying $\Sigma_j f_j g_j = 1$ and such that*

$$(1) \qquad \sum_j |g_j(z)|^2 \leqslant \delta^{-2} \quad \textit{for all } z \in D$$

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if and only if

$$\sum_j \|T_{f_j}^* u\|^2 \geq \delta^2 \|u\|^2 \quad \textit{for all } u \in H^2. \tag{2}$$

The statement of this Theorem differs from Theorem 6.3 of **[1]** only in that δ^{-2} appears in the inequality (1) where Arveson would have $16n^3\delta^{-6}$ when $\|f_i\|_\infty \leq 1$ for each i. It is easy to see that the estimate of (1) is sharp by choosing say $f_1 = 1, f_2 =, \ldots, = f_n = 0$. Our proof is quite distinct in that we deduce the theorem from a lifting theorem of Sz.-Nagy and Foias **[5]**.

PROOF OF THE THEOREM. Let H_n^2 be the product of n copies of H^2 with the usual product norm and inner product. Let A: $H_n^2 \to H^2$ be defined by $Au = \Sigma_j T_{f_j} u_j$, where $u = (u_1, \ldots, u_n) \in H_n^2$, and S as an operator on H_n^2 be defined by $S(u_1, \ldots, u_n) = (Su_1, \ldots, Su_n)$. Then $SA = AS$ and so $\mathfrak{R}(A)$, the range of A, is invariant under S. Also the conjugate of A, A^*, is given by $A^*v = (T_{f_1}^* v, \ldots, T_{f_n}^* v)$ and satisfies $S^*A^* = A^*S^*$.

Now suppose that (2) is satisfied; then $\|A^*v\|^2 \geq \delta^2\|v\|^2$ for all $v \in H^2$. In particular, $\mathfrak{R}(A^*)$ is a closed subspace of H_n^2 and on this subspace A^{*-1} exists and is bounded, $\|A^{*-1}\| \leq \delta^{-1}$ and $A^{*-1}S^* = SA^{*-1}$. Now S^* on H_n^2 is a co-isometric extension of S^* on $\mathfrak{R}(A^*)$ while S^* on H^2 is itself a co-isometry. Thus by the lifting theorem of **[5]** or Theorem 5 of **[4]**, there exists an operator J: $H_n^2 \to H^2$ such that $J'S^* = S^*J'$, $J'|\mathfrak{R}(A^*) = A^{*-1}$, and $\|J'\| = \|A^{*-1}\| \leq \delta^{-1}$. With $J = J'^*$, we then have that $JS = SJ$, $\|J\| \leq \delta^{-1}$ and $AJ = I$. Since any operator on H^2 which commutes with S is an analytic Toeplitz operator **[4]**, we deduce from the first of the above facts that there exist functions $g_1, \ldots, g_n \in H^\infty$ such that $Jv = (T_{g_1}v, \ldots, T_{g_n}v)$ for all $v \in H^2$. From the second fact we obtain

$$\sum_j |g_j(z)|^2 \leq \text{ess sup} \sum_j |g_j(e^{i\theta})|^2 = \|J\|^2 \leq \delta^{-2},$$

and finally from the identity $AJ = I$ we obtain

$$I = \sum_j T_{f_j} T_{g_j} = \sum_j T_{f_j g_j} = T_{\sum_j f_j g_j},$$

or $\Sigma_j f_j g_j = 1$.

Conversely, assuming that $g_1, \ldots, g_n \in H^\infty$ satisfy both $\Sigma_j f_j g_j = 1$ and (1), it is easy to prove (2) with $n^{-1}\delta^2$ in place of δ^2 by elementary manipulations with the identity $I = \Sigma_j T_{g_j}^* T_{f_j}^*$. To prove (2) as stated, more care is necessary. Let $Q = I - P$ (here I is the identity on L^2); then for all $u \in H^2$ and all $\varphi_1, \ldots, \varphi_n \in H^\infty$,

$$\begin{aligned}\sum_j \|T_{f_j}^* u\|^2 &= \sum_j \|PM_{\bar{f}_j} u\|^2 = \sum_j \|M_{\bar{f}_j} u\|^2 - \|QM_{\bar{f}_j} u\|^2 \\ &= \sum_j \|M_{\bar{f}_j} u\|^2 - \|QM_{\bar{f}_j + \varphi_j} u\|^2 \geq \sum_j \|M_{\bar{f}_j} u\|^2 - \|M_{\bar{f}_j + \bar{\varphi}_j} u\|^2.\end{aligned}$$

Thus

$$\sum_j \|T_{f_j}^* u\|^2 \geqslant \frac{1}{2\pi}\int_0^{2\pi}\left\{\sum_j |\bar{f}_j|^2 - |\bar{f}_j + \varphi_j|^2\right\}|u|^2\, d\theta. \tag{3}$$

Choosing $\varphi_j = -\delta^2 g_j$, where $g_1, \ldots, g_n \in H^2$ and satisfy (1), and $\sum_j f_j g_j = 1$, this inequality becomes

$$\sum_j \|T_{f_j}^* u\|^2 \geqslant \frac{1}{2\pi}\int_0^{2\pi}\left\{2\delta^2 - \delta^4 \sum_j |g_j|^2\right\}|u|^2\, d\theta$$

$$\geqslant \delta^2 \frac{1}{2\pi}\int_0^{2\pi}|u|^2\, d\theta \geqslant \delta^2\|u\|^2,$$

as required.

(3) can be written as $\sum_j \|T_{f_j}^* u\|^2 \geqslant \Delta^2\|u\|^2$ for all $u \in H^2$, where

$$\Delta^2 = \sup_{\varphi} \operatorname*{ess\,inf}_{|z|=1}\left\{\sum_j |f_j|^2 - |f_j + \bar{\varphi}_j|^2\right\}$$

and the supremum is taken over all $\varphi_1, \ldots, \varphi_n \in H^\infty$. If a positive lower bound for Δ could be found when $\sum_j |f_j|^2 \geqslant \varepsilon > 0$ and $f_1, \ldots, f_n \in H^\infty$, then the Theorem of this paper could be used to give another proof of the Corona Theorem. Estimation of Δ does not seem to be easy in any case of use for this purpose.

For $n = 1$, it is easy to establish that $\Delta^2 = \varepsilon$, for we need only put $\varphi = -\|f^{-1}\|_\infty^{-2} f$ in (3). For $n > 1$ the estimation of Δ can be changed into an approximation problem of independent interest. Choose $h, F_1, \ldots, F_n \in H^\infty$, h outer and such that $f_j = hF_j$ for $j = 1, \ldots, n$, and on $|z| = 1$, $\sum_j |f_j|^2 = |h|^2$ (this is possible). Then $\sum_j |F_j|^2 = 1$ on $|z| = 1$ and $|h|^2 \geqslant \varepsilon$ for all $z \in D$. Using the result just given for $n = 1$, and the preceding discussion, we have for all $u \in H^2$,

$$\sum_j \|T_{f_j}^* u\|^2 = \sum_j \|T_h^* T_{F_j}^* u\|^2 \geqslant \varepsilon \sum_j \|T_{F_j} u\|^2 \geqslant \varepsilon\sigma\|u\|^2,$$

where

$$\sigma = \sup_{\varphi} \operatorname*{ess\,inf}_{|z|=1}\left\{\sum_j |F_j|^2 - |F_j + \bar{\varphi}_j|^2\right\}$$

$$= 1 - \inf_{g} \operatorname*{ess\,sup}_{|z|=1} \sum_j |F_j - g_j|^2,$$

and the infimum is taken over all $g_1, \ldots, g_n$, such that $\bar{g}_1, \ldots, \bar{g}_n \in H^\infty$. Thus the estimation of σ and, hence, Δ becomes a question of estimating the best simultaneous approximation on $|z| = 1$ of $F_1, \ldots, F_n$ by functions $g_1, \ldots, g_n$ whose complex conjugates are in H^∞. Not only does such an estimate lead to a proof of the Corona Theorem, but also that theorem yields such an estimate.

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