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RING EXTENSIONS AND ESSENTIAL MONOMORPHISMS

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ABSTRACT. We study pairs of rings $R \subset S$ such that $\mathrm{Hom}_R(S, -)$: R-Mod $\to S$-Mod preserves essential monomorphisms. We obtain a complete characterization of such a pair in case S is a torsion-free algebra over a Noetherian domain $R \neq \mathrm{Quot}(R)$; S is then a left ideally finite R-algebra. The rings R such that every ring extension $R \subset S$ satisfies the above condition are subdirect sums of certain Artinian rings. Furthermore, we study a generalization of trivial ring extensions and show that the center of a semi-Artinian ring is again semi-Artinian.

Let $R \subset S$ be rings with $1_S \in R$. This paper is concerned with the question when the functor $\mathrm{Hom}_R(S, -)$: R-Mod $\to S$-Mod preserves essential monomorphisms or, equivalently, injective envelopes (condition (E)). D. Eisenbud has shown [1] that this is the case if S is finitely generated as an R-module by elements which centralize R, and an argument by Formanek and Jategaonkar [2] shows that the condition $Rs = sR$ for the generating elements s is sufficient; furthermore, the condition that ${}_RS$ be finitely generated can be weakened (Example 2). Another type of example is furnished by the construction $S = R \times M$, where M is an R-bimodule and multiplication is defined by an "associative" mapping $M \otimes_R M \to R$; Example 7 generalizes this situation. Our main result is Theorem 3: If R is a Noetherian domain but not a field and S a torsion-free R-algebra, then $R \subset S$ satisfies condition (E) if and only if S is an ideally finite R-algebra, i.e. every nonzero left S-module contains a nonzero submodule which is finitely generated over R. The proof of the theorem is essentially set-theoretical and the nontrivial part of it is false for non-Noetherian domains R. In Proposition 8 we give a short proof of the reflexion of relative injectivity and its consequence for chain conditions for (E)-extensions. We close with two examples involving semi-Artinian rings (Theorems 9, 10).

Finally, let us mention the corresponding problem defined by an R-module M and its endomorphism ring E. It is easy to see that, if M generates all of its submodules, then $\mathrm{Hom}_R(M, -)$: R-Mod $\to E$-Mod preserves essential monomorphisms. Under certain assumptions on M the converse to this fact has been shown in [8, Theorem 2.4]. Our case, however, where we consider rings $R \subset S$ (not $\mathrm{End}({}_RS)$) requires other methods since the multiplicative

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structure of S plays an important role besides its R-module structure.

All rings, modules, etc. are assumed to be unitary. For a subgroup U of some S-module M, denote by $(U: S)$ the largest S-submodule of M contained in U.

PROPOSITION 1. *For any pair of rings $R \subset S$ the following conditions are equivalent.*

(E) *The functor* $\mathrm{Hom}_R(S, -)$: R-Mod $\to$ S-Mod *preserves essential monomorphisms.*

(E′) *For every essential R-submodule U of a nonzero left S-module M, $(U: S)$ is not zero (resp. essential in ${}_SM$).*

PROOF. (E) $\Rightarrow$ (E′). Consider M as an S-submodule of $\mathrm{Hom}_R(S, M)$. Then $(U: S) = M \cap \mathrm{Hom}_R(S, U)$ is essential in M. (E′) $\Rightarrow$ (E). Since $(U: S) \cap N = (U \cap N: S)$ for every S-submodule N of M, both versions of (E′) are equivalent. Now let X be essential in ${}_RY$ and denote by U its inverse image under the mapping $\mathrm{Hom}_R(S, Y) \to Y$ defined by $f \mapsto f(1)$. Then $(U: S) = \mathrm{Hom}_R(S, X)$ which proves (E). $\square$

Let us call a pair of rings $R \subset S$ an (E)-*extension* if it satisfies the above conditions. Then, for any intermediate ring S' between R and S, $R \subset S'$ is also an (E)-extension.

A family $(J_\alpha)_{\alpha<\tau}$ of submodules of some module, indexed by an ordinal τ, is called a *continuous chain*, if it is increasing and if $J_\lambda = \bigcup_{\alpha<\lambda} J_\alpha$ for limit ordinals λ.

EXAMPLE 2. *Let $R \subset S$ be rings. Assume that ${}_SS_R$ contains a continuous chain of submodules $(J_\alpha)_{\alpha<\tau}$ terminating at $J_\tau = S$ such that, for all $\alpha < \tau$, $J_{\alpha+1}/J_\alpha$ is a sum of S-submodules which, as left R-modules, are generated by finitely many elements x_i satisfying $Rx_i = x_iR$. Then $R \subset S$ is an* (E)-*extension.*

PROOF. Let U be an essential R-submodule of some ${}_SM \neq (0)$. We may assume that the least ordinal α such that $J_\alpha M \neq (0)$ is 1. Then J_1 contains a left S-ideal $L = Rs_1 + \cdots + Rs_n$ such that $LM \neq (0)$ and $Rs_i = s_iR$ for all i. Say $s_1x \neq 0$ for some x in M. As in the proof of [2, Theorem 4], by induction on $j \leqslant n$, we can find elements $r_j \in R$ such that $(0) \neq (Rs_1 + \cdots + Rs_j)r_jx \subset U$. Hence $(0) \neq Lr_nx \subset (U: S)$. $\square$

Our main result is the converse to Example 2 in the following special case.

THEOREM 3. *Let R be a Noetherian domain different from its quotient field and A an R-algebra which is a torsion-free R-module. Then the following are equivalent.*

(a) *$R \subset A$ is an* (E)-*extension.* (b) *A contains a continuous chain of ideals $(J_\alpha)_{\alpha\leqslant\tau}$ terminating at $J_\tau = A$ such that each quotient $J_{\alpha+1}/J_\alpha$ is the sum of its left A-submodules which are finitely generated over R.*

The implication (a) $\Rightarrow$ (b) will follow from the more general Theorem 3′ below. For a domain $R \neq \mathrm{Quot}(R)$ let us denote by $i(R)$ the greatest cardinal such that every set of less than $i(R)$ nonzero ideals of R has nonzero intersection. Then $\aleph_0 \leqslant i(R) \leqslant \mathrm{card}(R)$, and $i(R) = \aleph_0$ when R is Noetherian. (See [4] for a domain R satisfying $i(R) = \aleph_1$.)

THEOREM 3′. *Let R be a domain different from its quotient field and $\mathfrak{a}$ a cardinal $\geqslant i(R)$. Further, let M and N be torsion-free R-modules and $(h_\alpha)_{\alpha \in \mathfrak{a}}$ a linearly independent family in* $\mathrm{Hom}_R(M, N)$ *such that every essential submodule of N contains the image of some nonzero mapping in $H = \sum_\alpha Rh_\alpha$. Then there exists a nonzero $h \in H$ of* $\mathrm{rank}(h) < \mathfrak{a}$.

We need two lemmas for the proof. Denote by $\mathfrak{a}$ an arbitrary infinite cardinal.

LEMMA 4. *Let $\{f_\alpha / \alpha \in \mathfrak{a}\} \subset$* $\mathrm{Hom}_K(V, W)$ *be a linearly independent family of homomorphisms of vector spaces over some field K such that every nonzero $f \in \sum_\alpha K f_\alpha$ has* $\mathrm{rank}(f) \geqslant \mathfrak{a}$. *Then there exists a family $\{X_\alpha / \alpha \in \mathfrak{a}\}$ of subspaces of V satisfying the following conditions.*

(a) *The sum $\sum_\alpha f_\alpha(X_\alpha)$ is direct.* (b) $X_\alpha \cap \mathrm{Ker}(f_\alpha) = (0)$ *for all α.* (c) $\dim_K(\bigcap X_{\alpha_i}) = \mathfrak{a}$ *for every finite subset $\{\alpha_i\} \subset \mathfrak{a}$.*

PROOF. We need the following fact from linear algebra. If finitely many $g_1, \ldots, g_m \in \mathrm{Hom}_K(U, W)$ are such that the vectors $g_1(u), \ldots, g_m(u)$ are linearly dependent for every $u \in U$, then there exist $k_i \in K$, not all zero, such that $k_1 g_1 + \cdots + k_m g_m$ has finite rank. Let us note the following consequence. If every nonzero linear combination of a linearly independent set $g_1, \ldots, g_m \in \mathrm{Hom}_K(V, W)$ is of rank $\geqslant \mathfrak{a}$, then every subspace U of V of codimension $< \mathfrak{a}$ contains a vector u such that $g_1(u), \ldots, g_m(u)$ are linearly independent.

We are now in a position to construct the spaces X_α. Denote by $\{F_\alpha / \alpha \in \mathfrak{a}\}$ the family of all finite nonempty subsets of $\mathfrak{a}$. By transfinite induction we can define a family $\{x_\alpha / \alpha \in \mathfrak{a}\} \subset V$ such that each family $\{f_\varepsilon(x_\alpha) / \varepsilon \in F_\alpha\}$ is linearly independent and that, denoting by S_α its linear span, the sum $\sum_\alpha S_\alpha$ is direct. To see that this definition is possible put $Y = \sum_{\alpha < \beta} S_\alpha$ for some $\beta \in \mathfrak{a}$ and set $W = Y \oplus C$ and $U = \bigcap_{\varepsilon \in F_\beta} f_\varepsilon^{-1}(C)$. Since $\dim_K(Y) < \mathfrak{a}$, by the preceding remark, we can find $x_\beta \in U$ such that $\{f_\varepsilon(x_\beta) / \varepsilon \in F_\beta\}$ is linearly independent and the sum $Y + S_\beta$ is direct. Finally, let $E_\varepsilon = \{\alpha \in \mathfrak{a} / \varepsilon \in F_\alpha\}$ and define $X_\varepsilon = \sum_{\alpha \in E_\varepsilon} K x_\alpha$ for all $\varepsilon \in \mathfrak{a}$. Properties (a), (b), (c) can now be easily checked. □

LEMMA 5. *Let $R \neq$* $\mathrm{Quot}(R)$ *be a domain and F a torsion-free R-module of* $\mathrm{rank}(F) = \mathfrak{a} \geqslant i(R)$. *Suppose further a set Φ of submodules of F to be given, of cardinality $\leqslant \mathfrak{a}$, such that every essential submodule of F contains some rU where $U \in \Phi$ and $r \in R - \{0\}$. Then Φ must contain a module U of* $\mathrm{rank}(U) < \mathfrak{a}$.

PROOF. Let $\Phi = \{U_\alpha / \alpha \in \mathfrak{a}\}$. By a routine reduction argument, we may restrict ourselves to the case when $F = \bigoplus_\alpha R x_\alpha$ is free on the basis $\{x_\alpha / \alpha \in$

$\mathfrak{a}\}$ and $U_\alpha = \bigoplus_\beta R s_{\alpha\beta} x_\beta$ with $s_{\alpha\beta} \in R$. Then let $T_\alpha = \{\beta \in \mathfrak{a}/s_{\alpha\beta} \neq 0\}$ and assume, by way of contradiction, that $\operatorname{rank}(U_\alpha) = \operatorname{card}(T_\alpha) = \mathfrak{a}$ for all $\alpha \in \mathfrak{a}$. By a set-theoretical argument we can find subsets $S_\alpha \subset T_\alpha$ such that $\operatorname{card}(S_\alpha) = \mathfrak{a}$ and $S_\alpha \cap S_\beta = \varnothing$ for all $\alpha \neq \beta$. Since $i(R) \leqslant \mathfrak{a}$, there is a subset $\{r_\alpha/\alpha \in \mathfrak{a}\} \subset R - \{0\}$ such that $\bigcap_\alpha Rr_\alpha = (0)$. Put $r_{\alpha\beta} = r_{j_\alpha(\beta)}$ where j_α: $S_\alpha \to \mathfrak{a}$ is some bijective mapping and set $E_\alpha = \bigoplus_{\beta \in S_\alpha} R r_{\alpha\beta} s_{\alpha\beta} x_\beta$. Then the direct sum $E = \bigoplus_\alpha E_\alpha + \bigoplus_{\gamma \in \mathfrak{b}} R x_\gamma$ with $\mathfrak{b} = \mathfrak{a} - \bigcup_\alpha S_\alpha$ is essential in F. Hence there exist $\delta \in \mathfrak{a}$ and $r \neq 0$ such that $rU_\delta \subset E$. Comparing coefficients now yields the contradiction $r \in \bigcap_{\beta \in S_\delta} R r_{\delta\beta} = (0)$. □

Proof of Theorem 3′. Suppose that $\operatorname{rank}(h) \geqslant \mathfrak{a}$ for every nonzero $h \in H$. Then let $K = \operatorname{Quot}(R)$ and put $V = K \otimes M$, $W = K \otimes N$, and $f_\alpha = 1 \otimes h_\alpha$. Let (X_α) be the family of subspaces of V corresponding to the f_α's as described in Lemma 4 and set $Y_\alpha = M \cap X_\alpha$ and $F_\alpha = h_\alpha(Y_\alpha)$. By Lemma 4, the submodule $h_\alpha(Y_\alpha \cap \bigcap_i Y_{\alpha_i})$ of F_α has rank $\mathfrak{a}$ for every finite subset $\{\alpha_i\}$ of $\mathfrak{a}$. Denote by Φ_α the set of all such submodules of F_α. By Lemma 5, F_α must contain an essential submodule E_α not containing any of the modules rU with $U \in \Phi_\alpha$ and $r \neq 0$. Choose a submodule C of N such that the sum $E = \bigoplus_\alpha E_\alpha + C$ is direct and essential in N and let $h = r_1 h_{\alpha_1} + \cdots + r_n h_{\alpha_n}$ with nonzero $r_i \in R$ and different α_i be such that $h(M) \subset E$. Then $h(\bigcap_i Y_{\alpha_i}) \subset E \cap (\bigoplus_\alpha F_\alpha) = \bigoplus_\alpha E_\alpha$ and, hence, $r_1 h_{\alpha_1}(\bigcap_i Y_{\alpha_i}) \subset E_{\alpha_1}$, contradicting the choice of E_{α_1}. □

Corollary 6. *Let the domain R satisfy $i(R) = \aleph_0$. Let N be a nonzero left module over some R-algebra A such that ${}_RN$ is torsion-free and $(U: A)$ is an essential A-submodule for every essential R-submodule U of N. Then N contains a finitely generated R-submodule F such that $(F: A) \neq (0)$.*

Proof. Consider a nonzero A-submodule X of N of least R-rank and apply Theorem 3′ to homomorphisms of the form $a \mapsto ax$ for $a \in A$ and $x \in X$. □

Proof of Theorem 3. For a left A-module M, denote by $q(M)$ the sum of its submodules which are finitely generated over R. Defining $J_{\alpha+1}/J_\alpha = q(A/J_\alpha)$, Corollary 6 shows that $J_\tau = A$ for some ordinal τ. □

Remarks. (1) Following [7], let us call an algebra A over an arbitrary commutative ring R *left ideally finite* if it satisfies condition (b) of Theorem 3, or, equivalently, if any nonzero left A-module contains a nonzero submodule which is finitely generated over R. Such an algebra is easily seen to be locally finite.

(2) For a non-Noetherian domain R, Theorem 3, (a) $\Rightarrow$ (b), is false, in general, as can be seen by taking the polynomial ring $A = \mathbf{Z}[X]$ and its subring $R = \mathbf{Z} + pA$ for some prime p. For an essential R-submodule U of some ${}_AM \neq (0)$, we have $(0) \neq N + JU \subset (U: A)$, where N denotes the annihilator of $J = pA$ in M. But A is not integral over R. □

Example 7. *Let $R \subset S$ be rings and assume that S_R contains a submodule J such that $S = R + J$ and, given any sequence $s_0, s_1, \ldots,$ of elements of J, there* *is an index n such that $s_n s_{n-1} \cdots s_0 \in R$. Then $R \subset S$ has property* (E) *and every nonzero left S-module M contains a finitely generated R-submodule F such that* $(F: S) \neq (0)$.

PROOF. Suppose $(U: S) = (0)$ for an essential R-submodule U of some ${}_SM \neq (0)$. Then $Ju \not\subset U$ for every nonzero $u \in U$. Fixing such an element u and using the fact that U is R-essential we obtain a sequence $s_0, s_1, \ldots$ from J such that $s_n s_{n-1} \cdots s_0 u \notin U$ for all n, a contradiction. To prove the second statement assume $(F: S) = (0)$ for every f.g. ${}_RF$ in M and let $x \in M - \{0\}$. In a way similar to the preceding argument we can find elements $s_n \in J$ and f.g. R-submodules F_n such that $x \in F_n$ and $s_n s_{n-1} \cdots s_0 x \notin F_n$ for all n, which gives the contradiction. □

REMARK. Let $R \subset S$ be as above and assume that, in addition, J is an ideal in S. Then, as for trivial ring extensions, every injective left S-module can be shown to be isomorphic to $\mathrm{Hom}_R(S, X)$ for some ${}_RX$. □

The results of [**1**] and [**2**] on descent of chain condition for ring extensions of the type of Example 2 apply also to those described in Example 7. Let us mention the following footnote to Eisenbud's paper.

PROPOSITION 8. *Let $R \subset S$ be rings.* (a) *If condition* (E) *holds and if ${}_SM$ and ${}_RX$ are such that* $\mathrm{Hom}_R(S,X)$ *is M-injective, then X is ${}_RM$-injective* (*condition* (I)). (b) *Assume that condition* (I) *holds and that every nonzero ${}_SM$ contains some f.g. R-submodule F such that* $(F: S) \neq (0)$. *Then any Noetherian left S-module is Noetherian as an R-module.*

PROOF. (a) Recall a module X being Y-injective if every homomorphism $Z \to X$, Z a submodule of Y, can be extended to Y. Since this is the case if and only if $\mathrm{Hom}(Y, X) \to \mathrm{Hom}(Y, E(X))$ is an isomorphism for $E(X)$ an injective hull of X, the statement follows by a straightforward adjointness argument. (b) Let ${}_SM$ be Noetherian. From the second part of the assumption it follows that ${}_RM$ is finitely generated (consider a maximal $(F: S)$ with ${}_RF$ f.g.). By a well-known argument due to Bass it suffices to show ${}_RM$ to be X-injective for X an arbitrary direct sum of injective R-modules. But $\mathrm{Hom}_R(S, X)$ is M-injective since every submodule of ${}_SM$ is f.g. over R. Thus condition (I) yields the conclusion. □

In closing, let us note two examples of (E)-extensions of semi-Artinian rings. The following statement can be expressed by saying that the pair $R \subset S$ in question has property (E).

THEOREM 9. *The center R of a left semi-Artinian ring S is semi-Artinian.*

PROOF. Let x be a central element of an arbitrary ring S with left socle I. If $Sx^n \equiv Sx^{n+1} \pmod{I}$ for some $n \geqslant 0$, then $Sx^m = Sx^{m+1}$ for some $m \geqslant 0$. To see this, let $s \in S$ such that $y = x^n - sx^{n+1} \in I$. Since Sy is Artinian, we get $Sy^k = Sy^{k+1}$ for some $k \geqslant 1$. Centrality of x now yields $x^{nk} \in Sx^{nk+1}$, so $m = nk$ does it.

Next we claim that $Sx^n = Sx^{n+1}$ for every $x \in R$ and some n depending on x. Let $(I_\alpha)_{\alpha<\tau}$ be the Loewy series of ${}_SS$, i.e. the continuous chain of ideals defined by $I_\tau = S$ and $I_{\alpha+1}/I_\alpha = \mathrm{soc}({}_SS/I_\alpha)$. We must show that the least ordinal α such that $Sx^n \equiv Sx^{n+1} \pmod{I_\alpha}$ for some $n > 0$ is zero. If not, $\alpha = \beta + 1$. Then consider the ring $S' = S/I_\beta$, $x' = x + I_\beta$, and $I' = I_\alpha/I_\beta$. By the preceding remark it follows that $S'x'^m = S'x'^{m+1}$ for some $m \geqslant 0$, contradicting the minimality of α. It follows that every $x \in R$ satisfies some equation $Rx^n = Rx^{n+1}$ [5, Satz 2.5], i.e. R has Krull dimension zero. Hence every maximal left ideal of S has maximal intersection with R. Thus S is a semi-Artinian R-module. Q.E.D.

THEOREM 10. *For any ring R the following conditions are equivalent.*

(a) *Every ring extension $R \hookrightarrow S$ has property* (E).

(b) *$R/t(R)$, $t(R)$ being the torsion part of $(R, +)$, and R/pR, for every prime p, are semisimple rings.*

(c) *$R = A \times B$, with B a semisimple* **Q**-*algebra and A a subring of the product $\prod_p A_p$ of Artinian rings A_p satisfying* $\mathrm{rad}(A_p) = pA_p$ *for every prime p, such that A contains the ideal $I = \bigoplus_p A_p$ and A/I is semisimple.*

PROOF. (a) $\Rightarrow$ (b) For any homomorphism of rings $h\colon R \to T$ and simple left T-module, the induced R-module $M_{(h)}$ is semisimple. This can be seen by making M a simple module over $S = R \times T$ and applying Proposition 8(a) to $R \hookrightarrow S$. Thus, any left R-module whose ring of endomorphisms contains a subfield must be semisimple. In particular, so are $\mathbf{Q} \otimes_{\mathbf{Z}} R/t(R)$ and R/pR. (c) $\Rightarrow$ (b) $\Rightarrow$ (a) is easy.

(b) $\Rightarrow$ (c) Denote by $t_p(R)$ the p-component and by $d(R)$ the divisible part of the additive group of R. Since $R/t(R)$ is divisible, the argument from [3, Lemma 2] yields $R = p^nR \oplus t_p(R)$ for some $n \geqslant 0$. Setting $B = d(R)$ we have $B \cap t(R) = (0)$ since all components of R are bounded. Thus B is a module over $R/t(R)$. These modules are easily seen to be injective over R. Thus $R = B \oplus A$ with A a left ideal. A is also a right ideal because the right annihilator of B in R has zero intersection with B. Finally, consider the unitary ring $A_p = t_p(R)$. Since it is bounded as a group and since $A_p/pA_p \cong R/pR$ is semisimple, it is Artinian with radical pA_p. The remaining part is obvious. □

REMARK. Let A be an Artinian ring such that $\mathrm{rad}(A) = pA$ for some prime p. Then A is the product of a finite number of full matrix rings over local Artinian rings A' satisfying $\mathrm{rad}(A') = pA'$. For more information about these rings A', see [6].

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