RING EXTENSIONS AND ESSENTIAL MONOMORPHISMS

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Abstract. We study pairs of rings $R \subset S$ such that $\text{Hom}_R(S, -) : R\text{-Mod} \rightarrow S\text{-Mod}$ preserves essential monomorphisms. We obtain a complete characterization of such a pair in case $S$ is a torsion-free algebra over a Noetherian domain $R \neq \text{Quot}(R)$; $S$ is then a left ideally finite $R$-algebra. The rings $R$ such that every ring extension $R \subset S$ satisfies the above condition are subdirect sums of certain Artinian rings. Furthermore, we study a generalization of trivial ring extensions and show that the center of a semi-Artinian ring is again semi-Artinian.

Let $R \subset S$ be rings with $1_S \in R$. This paper is concerned with the question when the functor $\text{Hom}_R(S, -) : R\text{-Mod} \rightarrow S\text{-Mod}$ preserves essential monomorphisms or, equivalently, injective envelopes (condition (E)). D. Eisenbud has shown [1] that this is the case if $S$ is finitely generated as an $R$-module by elements which centralize $R$, and an argument by Formanek and Jategaonkar [2] shows that the condition $Rs = sR$ for the generating elements $s$ is sufficient; furthermore, the condition that $Rs$ be finitely generated can be weakened (Example 2). Another type of example is furnished by the construction $S = R \times M$, where $M$ is an $R$-bimodule and multiplication is defined by an “associative” mapping $M \otimes_R M \rightarrow R$; Example 7 generalizes this situation. Our main result is Theorem 3: If $R$ is a Noetherian domain but not a field and $S$ a torsion-free $R$-algebra, then $R \subset S$ satisfies condition (E) if and only if $S$ is an ideally finite $R$-algebra, i.e. every nonzero left $S$-module contains a nonzero submodule which is finitely generated over $R$. The proof of the theorem is essentially set-theoretical and the nontrivial part of it is false for non-Noetherian domains $R$. In Proposition 8 we give a short proof of the reflexivity of relative injectivity and its consequence for chain conditions for (E)-extensions. We close with two examples involving semi-Artinian rings (Theorems 9, 10).

Finally, let us mention the corresponding problem defined by an $R$-module $M$ and its endomorphism ring $E$. It is easy to see that, if $M$ generates all of its submodules, then $\text{Hom}_R(M, -) : R\text{-Mod} \rightarrow E\text{-Mod}$ preserves essential monomorphisms. Under certain assumptions on $M$ the converse to this fact has been shown in [8, Theorem 2.4]. Our case, however, where we consider rings $R \subset S$ (not $\text{End}(R)$) requires other methods since the multiplicative
structure of \( S \) plays an important role besides its \( R \)-module structure.

All rings, modules, etc. are assumed to be unitary. For a subgroup \( U \) of some \( S \)-module \( M \), denote by \((U: S)\) the largest \( S \)-submodule of \( M \) contained in \( U \).

**Proposition 1.** For any pair of rings \( R \subset S \) the following conditions are equivalent.

(E) The functor \( \text{Hom}_R(S, -) : R\text{-Mod} \to S\text{-Mod} \) preserves essential monomorphisms.

(E') For every essential \( R \)-submodule \( U \) of a nonzero left \( S \)-module \( M \), \((U: S)\) is not zero (resp. essential in \( SM \)).

**Proof.** (E) \( \Rightarrow \) (E'). Consider \( M \) as an \( S \)-submodule of \( \text{Hom}_S(S, M) \). Then 
\[
(U : S) = M \cap \text{Hom}_S(S, U)
\]
is essential in \( M \). (E') \( \Rightarrow \) (E). Since \((U: S) \cap N = (U \cap N : S)\) for every \( S \)-submodule \( N \) of \( M \), both versions of (E') are equivalent. Now let \( X \) be essential in \( Y \) and denote by \( U \) its inverse image under the mapping \( \text{Hom}_R(S, Y) \to Y \) defined by \( f \mapsto f(1) \). Then \((U : S) = \text{Hom}_R(S, X)\) which proves (E). □

Let us call a pair of rings \( R \subset S \) an (E)-extension if it satisfies the above conditions. Then, for any intermediate ring \( S' \) between \( R \) and \( S \), \( R \subset S' \) is also an (E)-extension.

A family \( (J_\alpha)_{\alpha \leq \tau} \) of submodules of some module, indexed by an ordinal \( \tau \), is called a continuous chain, if it is increasing and if \( J_\lambda = \bigcup_{\alpha < \lambda} J_\alpha \) for limit ordinals \( \lambda \).

**Example 2.** Let \( R \subset S \) be rings. Assume that \( S = \bigcup_{\alpha < \tau} J_\alpha \) contains a continuous chain of submodules \( (J_\alpha)_{\alpha < \tau} \) terminating at \( J_\tau = S \) such that, for all \( \alpha < \tau \), \( J_{\alpha+1}/J_\alpha \) is a sum of \( S \)-submodules which, as left \( R \)-modules, are generated by finitely many elements \( x_i \) satisfying \( Rx_i = x_i R \). Then \( R \subset S \) is an (E)-extension.

**Proof.** Let \( U \) be an essential \( R \)-submodule of some \( SM \neq (0) \). We may assume that the least ordinal \( \alpha \) such that \( J_\alpha M \neq (0) \) is 1. Then \( J_1 \) contains a left \( S \)-ideal \( L = R s_1 + \cdots + R s_n \) such that \( LM \neq (0) \) and \( R s_i = s_i R \) for all \( i \). Say \( s_1 x \neq 0 \) for some \( x \) in \( M \). As in the proof of [2, Theorem 4], by induction on \( j \leq n \), we can find elements \( r_j \in R \) such that \((0) \neq (R s_1 + \cdots + R s_j)r_j x \subset U \). Hence \((0) \neq L r_n x \subset (U : S) \). □

Our main result is the converse to Example 2 in the following special case.

**Theorem 3.** Let \( R \) be a Noetherian domain different from its quotient field and \( A \) an \( R \)-algebra which is a torsion-free \( R \)-module. Then the following are equivalent.

(a) \( R \subset A \) is an (E)-extension. (b) \( A \) contains a continuous chain of ideals \( (J_\alpha)_{\alpha \leq \tau} \) terminating at \( J_\tau = A \) such that each quotient \( J_{\alpha+1}/J_\alpha \) is the sum of its left \( A \)-submodules which are finitely generated over \( R \).

The implication (a) \( \Rightarrow \) (b) will follow from the more general Theorem 3' below. For a domain \( R \neq \text{Quot}(R) \) let us denote by \( i(R) \) the greatest cardinal.
such that every set of less than \( i(R) \) nonzero ideals of \( R \) has nonzero intersection. Then \( n_0 < i(R) < \text{card}(R) \), and \( i(R) = n_0 \) when \( R \) is Noetherian. (See [4] for a domain \( R \) satisfying \( i(R) = n_1 \).)

**Theorem 3'.** Let \( R \) be a domain different from its quotient field and \( \alpha \) a cardinal \( > i(R) \). Further, let \( M \) and \( N \) be torsion-free \( R \)-modules and \( (h_\alpha)_{\alpha \in \alpha} \) a linearly independent family in \( \text{Hom}_R(M, N) \) such that every essential submodule of \( N \) contains the image of some nonzero mapping in \( H = \sum_\alpha Rh_\alpha \). Then there exists a nonzero \( h \in H \) of rank(\( h \)) < \( \alpha \).

We need two lemmas for the proof. Denote by \( \alpha \) an arbitrary infinite cardinal.

**Lemma 4.** Let \( \{f_\alpha/\alpha \in \alpha\} \subset \text{Hom}_K(V, W) \) be a linearly independent family of homomorphisms of vector spaces over some field \( K \) such that every nonzero \( f \in \Sigma_\alpha Kf_\alpha \) has rank(\( f \)) \( > \alpha \). Then there exists a family \( \{X_\alpha/\alpha \in \alpha\} \) of subspaces of \( V \) satisfying the following conditions.

(a) The sum \( \Sigma_\alpha f_\alpha(X_\alpha) \) is direct. (b) \( X_\alpha \cap \text{Ker}(f_\alpha) = (0) \) for all \( \alpha \). (c) \( \dim_K(\cap X_\alpha) = \alpha \) for every finite subset \( \{\alpha_\beta\} \subset \alpha \).

**Proof.** We need the following fact from linear algebra. If finitely many \( g_1, \ldots, g_m \in \text{Hom}_K(U, W) \) are such that the vectors \( g_1(u), \ldots, g_m(u) \) are linearly dependent for every \( u \in U \), then there exist \( k_1 \in K \), not all zero, such that \( k_1 g_1 + \cdots + k_m g_m \) has finite rank. Let us note the following consequence. If every nonzero linear combination of a linearly independent set \( g_1, \ldots, g_m \in \text{Hom}_K(V, W) \) is of rank \( \geq \alpha \), then every subspace \( U \) of \( V \) of codimension \( < \alpha \) contains a vector \( u \) such that \( g_1(u), \ldots, g_m(u) \) are linearly independent.

We are now in a position to construct the spaces \( X_\alpha \). Denote by \( \{F_\alpha/\alpha \in \alpha\} \) the family of all finite nonempty subsets of \( \alpha \). By transfinite induction we can define a family \( \{x_\alpha/\alpha \in \alpha\} \subset V \) such that each family \( \{f_\epsilon(x_\alpha)/\epsilon \in F_\alpha\} \) is linearly independent and that, denoting by \( S_\alpha \) its linear span, the sum \( \Sigma_\alpha S_\alpha \) is direct. To see that this definition is possible put \( Y = \Sigma_\alpha < \beta S_\alpha \) for some \( \beta \in \alpha \) and set \( W = Y \oplus C \) and \( U = \cap \epsilon \in F_\alpha f_\epsilon^{-1}(C) \). Since \( \dim_K(Y) < \alpha \), by the preceding remark, we can find \( x_\beta \in U \) such that \( \{f_\epsilon(x_\beta)/\epsilon \in F_\beta\} \) is linearly independent and the sum \( Y + S_\beta \) is direct. Finally, let \( E_\epsilon = \{\alpha \in \alpha/\epsilon \in F_\alpha\} \) and define \( X_\alpha = \Sigma_\alpha E_\alpha Kx_\alpha \) for all \( \epsilon \in \alpha \). Properties (a), (b), (c) can now be easily checked. □

**Lemma 5.** Let \( R \neq \text{Quot}(R) \) be a domain and \( F \) a torsion-free \( R \)-module of rank(\( F \)) \( = \alpha \geq i(R) \). Suppose further a set \( \Phi \) of submodules of \( F \) to be given, of cardinality \( < \alpha \), such that every essential submodule of \( F \) contains some \( rU \) where \( U \in \Phi \) and \( r \in R - \{0\} \). Then \( \Phi \) must contain a module \( U \) of rank(\( U \)) \( < \alpha \).

**Proof.** Let \( \Phi = \{U_\alpha/\alpha \in \alpha\} \). By a routine reduction argument, we may restrict ourselves to the case when \( F = \bigoplus_\alpha Rx_\alpha \) is free on the basis \( \{x_\alpha/\alpha \in \alpha\} \).
a) and \( \mathcal{U}_a = \bigoplus \beta R_s \beta x_\beta \) with \( s_\beta \in R \). Then let \( T_a = \{ \beta \in a / s_\beta \neq 0 \} \) and assume, by way of contradiction, that \( \text{rank}(\mathcal{U}_a) = \text{card}(T_a) = a \) for all \( a \in a \).

By a set-theoretical argument we can find subsets \( S_a \subset T_a \) such that \( \text{card}(S_a) = a \) and \( S_a \cap S_\beta = \emptyset \) for all \( a \neq \beta \). Since \( i(R) \leq a \), there is a subset \( \{ r_\alpha / \alpha \in a \} \subset R - \{ 0 \} \) such that \( \cap_\alpha R_{r_\alpha} = \{ 0 \} \). Put \( r_{a\beta} = r_{a_\beta} \left( \right) \) where \( j_a: S_a \to a \) is some bijective mapping and set \( E_a = \bigoplus \beta \in S_a R \beta \alpha_\beta x_\beta \). Then the direct sum \( E = \bigoplus_{a \in a} E_a + \bigoplus_{\gamma \in \gamma} R_x \gamma \) with \( \beta = b - \cup_\alpha S_a \) is essential in \( F \).

Hence there exist \( \delta \in a \) and \( r \neq 0 \) such that \( rU_\delta \subset E \). Comparing coefficients now yields the contradiction \( r \in \cap_\beta \cup_\alpha S_\beta \delta = \{ 0 \} \).

**Proof of Theorem 3'.** Suppose that \( \text{rank}(h) > a \) for every nonzero \( h \in H \). Then let \( K = \text{Quot}(R) \) and put \( V = K \otimes M, W = K \otimes N, \) and \( f_a = 1 \otimes h_a \).

Let \( (X_a) \) be the family of subspaces of \( V \) corresponding to the \( f_a \)'s as described in Lemma 4 and set \( Y_a = M \cap X_a \) and \( F_a = h_a(Y_a) \). By Lemma 4, the submodule \( h_a(Y_a \cap \cap Y_a) \) of \( F_a \) has rank \( a \) for every finite subset \( \{ a_\alpha \} \) of \( a \). Denote by \( \Phi_a \) the set of all such submodules of \( F_a \). By Lemma 5, \( F_a \) must contain an essential submodule \( E_a \) not containing any of the modules \( rU \) with \( U \in \Phi_a \) and \( r \neq 0 \). Choose a submodule \( C \) of \( N \) such that the sum \( E = \bigoplus_a E_a + C \) is direct and essential in \( N \) and let \( h = r_1 h_{a_1} + \cdots + r_n h_{a_n} \) with nonzero \( r_i \in R \) and different \( a_i \) be such that \( h(M) \subset E \). Then \( \cap (Y_a) \subset E \cap (\bigoplus_a F_a) = \bigoplus_a E_a \) and, hence, \( r_1 h_{a_1}(\cap, Y_a) \in E_{a_1} \), contradicting the choice of \( E_{a_1} \).

**Corollary 6.** Let the domain \( R \) satisfy \( i(R) = \mathfrak{N}_0 \). Let \( N \) be a nonzero left module over some \( R \)-algebra \( A \) such that \( R \) is torsion-free and \( (U: A) \) is an essential \( A \)-submodule for every essential \( R \)-submodule \( U \) of \( N \). Then \( N \) contains a finitely generated \( R \)-submodule \( F \) such that \( (F: A) \neq \{ 0 \} \).

**Proof.** Consider a nonzero \( A \)-submodule \( X \) of \( N \) of least \( \mathfrak{N} \)-rank and apply Theorem 3' to homomorphisms of the form \( a \mapsto ax \) for \( a \in A \) and \( x \in X \).

**Proof of Theorem 3.** For a left \( A \)-module \( M \), denote by \( q(M) \) the sum of its submodules which are finitely generated over \( R \). Defining \( J_{a+1} / J_a = q(A/J_a) \), Corollary 6 shows that \( J_{\tau} = A \) for some ordinal \( \tau \).

**Remarks.** (1) Following [7], let us call an algebra \( A \) over an arbitrary commutative ring \( R \) left ideally finite if it satisfies condition (b) of Theorem 3, or, equivalently, if any nonzero left \( A \)-module contains a nonzero submodule which is finitely generated over \( R \). Such an algebra is easily seen to be locally finite.

(2) For a non-Noetherian domain \( R \), Theorem 3, (a) \( \Rightarrow \) (b), is false, in general, as can be seen by taking the polynomial ring \( A = \mathbb{Z}[X] \) and its subring \( R = \mathbb{Z} + pA \) for some prime \( p \). For an essential \( R \)-submodule \( U \) of some \( \hat{M} \neq (0) \), we have \( (0) \neq M + JU \subset (U: A) \), where \( N \) denotes the annihilator of \( J = pA \) in \( M \). But \( A \) is not integral over \( R \).

**Example 7.** Let \( R \subset S \) be rings and assume that \( S_R \) contains a submodule \( J \) such that \( S = R + J \) and, given any sequence \( s_0, s_1, \ldots \), of elements of \( J \), there
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is an index \(n\) such that \(s_n s_{n-1} \cdots s_0 \in R\). Then \(R \subseteq S\) has property \((E)\) and every nonzero left \(S\)-module \(M\) contains a finitely generated \(R\)-submodule \(F\) such that \((F: S) \neq (0)\).

**Proof.** Suppose \((U: S) = (0)\) for an essential \(R\)-submodule \(U\) of some \(S\)-module \(M \neq (0)\). Then \(Ju \nsubseteq U\) for every nonzero \(u \in U\). Fixing such an element \(u\) and using the fact that \(U\) is \(R\)-essential we obtain a sequence \(s_0, s_1, \ldots\) from \(J\) such that \(s_n s_{n-1} \cdots s_{0} u \notin U\) for all \(n\), a contradiction. To prove the second statement assume \((F: S) = (0)\) for every f.g. \(R\)-module \(F\) in \(M\) and let \(x \in M - \{0\}\). In a way similar to the preceding argument we can find elements \(s_n \in J\) and f.g. \(R\)-submodules \(F_n\) such that \(x \in F_n\) and \(s_n s_{n-1} \cdots s_0 x \not\in F_n\) for all \(n\), which gives the contradiction. \(\square\)

**Remark.** Let \(R \subseteq S\) be as above and assume that, in addition, \(J\) is an ideal in \(S\). Then, as for trivial ring extensions, every injective left \(S\)-module can be shown to be isomorphic to \(\text{Hom}_R(S, X)\) for some \(R\)-module \(X\). \(\square\)

The results of [1] and [2] on descent of chain condition for ring extensions of the type of Example 2 apply also to those described in Example 7. Let us mention the following footnote to Eisenbud's paper.

**Proposition 8.** Let \(R \subseteq S\) be rings. (a) If condition \((E)\) holds and if \(S\)-module and \(R\)-module are such that \(\text{Hom}_R(S, X)\) is \(M\)-injective, then \(X\) is \(R\)-module-injective (condition \((I)\)). (b) Assume that condition \((I)\) holds and that every nonzero \(S\)-module contains some f.g. \(R\)-submodule \(F\) such that \((F: S) \neq (0)\). Then any Noetherian left \(S\)-module is Noetherian as an \(R\)-module.

**Proof.** (a) Recall a module \(X\) being \(Y\)-injective if every homomorphism \(Z \rightarrow X\), \(Z\) a submodule of \(Y\), can be extended to \(Y\). Since this is the case if and only if \(\text{Hom}(Y, X) \rightarrow \text{Hom}(Y, E(X))\) is an isomorphism for \(E(X)\) an injective hull of \(X\), the statement follows by a straightforward adjointness argument. (b) Let \(S\)-module be Noetherian. From the second part of the assumption it follows that \(R\)-module is finitely generated (consider a maximal \((F: S)\) with \(R\)-module f.g.). By a well-known argument due to Bass it suffices to show \(R\)-module to be \(X\)-injective for \(X\) an arbitrary direct sum of injective \(R\)-modules. But \(\text{Hom}_R(S, X)\) is \(M\)-injective since every submodule of \(S\)-module is f.g. over \(R\). Thus condition \((I)\) yields the conclusion. \(\square\)

In closing, let us note two examples of \((E)\)-extensions of semi-Artinian rings. The following statement can be expressed by saying that the pair \(R \subseteq S\) in question has property \((E)\).

**Theorem 9.** The center \(R\) of a left semi-Artinian ring \(S\) is semi-Artinian.

**Proof.** Let \(x\) be a central element of an arbitrary ring \(S\) with left socle \(I\). If \(Sx^n \equiv Sx^{n+1} \pmod{I}\) for some \(n > 0\), then \(Sx^m = Sx^{m+1}\) for some \(m > 0\). To see this, let \(s \in S\) such that \(y = x^n - sx^{n+1} \in I\). Since \(S\)-module is Artinian, we get \(Sx^k = Sx^{k+1}\) for some \(k > 1\). Centrality of \(x\) now yields \(x^{nk} \in Sx^{nk+1}\), so \(m = nk\) does it.

Next we claim that \(Sx^n = Sx^{n+1}\) for every \(x \in R\) and some \(n\) depending
on $x$. Let $(I_a)_{a<\alpha}$ be the Loewy series of $SS$, i.e. the continuous chain of ideals defined by $I_r = S$ and $I_{a+1}/I_a = \text{soc}(S^\infty/I_a)$. We must show that the least ordinal $\alpha$ such that $S^\infty x^\alpha \equiv S^\infty x^{\alpha+1} \pmod {I_\alpha}$ for some $n > 0$ is zero. If not, $\alpha = \beta + 1$. Then consider the ring $S' = S/I_\beta$, $x' = x + I_\beta$, and $I' = I_a/I_\beta$. By the preceding remark it follows that $S' x' = S' x'^{m+1}$ for some $m > 0$, contradicting the minimality of $\alpha$. It follows that every $x \in R$ satisfies some equation $Rx^n = Rx^{n+1}$ [5, Satz 2.5], i.e. $R$ has Krull dimension zero. Hence every maximal left ideal of $S$ has maximal intersection with $R$. Thus $S$ is a semi-Artinian $R$-module. Q.E.D.

**Theorem 10.** For any ring $R$ the following conditions are equivalent.

(a) Every ring extension $R \to S$ has property (E).

(b) $R/t(R)$, $t(R)$ being the torsion part of $(R, +)$, and $R/pR$, for every prime $p$, are semisimple rings.

(c) $R = A \times B$, with $B$ a semisimple $\mathbb{Q}$-algebra and $A$ a subring of the product $\prod_p A_p$ of Artinian rings $A_p$ satisfying $\text{rad}(A_p) = pA_p$ for every prime $p$, such that $A$ contains the ideal $I = \bigoplus_p A_p$ and $A/I$ is semisimple.

**Proof.** (a) $\Rightarrow$ (b) For any homomorphism of rings $h: R \to T$ and simple left $T$-module, the induced $R$-module $M(h)$ is semisimple. This can be seen by making $M$ a simple module over $S = R \times T$ and applying Proposition 8(a) to $R \to S$. Thus, any left $R$-module whose ring of endomorphisms contains a subfield must be semisimple. In particular, so are $\mathbb{Q} \otimes \mathbb{Z} R/t(R)$ and $R/pR$.

(c) $\Rightarrow$ (b) $\Rightarrow$ (a) is easy.

(b) $\Rightarrow$ (c) Denote by $t_p(R)$ the $p$-component and by $d(R)$ the divisible part of the additive group of $R$. Since $R/t(R)$ is divisible, the argument from [3, Lemma 2] yields $R = p^n R \oplus t_p(R)$ for some $n > 0$. Setting $B = d(R)$ we have $B \cap t(R) = (0)$ since all components of $R$ are bounded. Thus $B$ is a module over $R/t(R)$. These modules are easily seen to be injective over $R$. Thus $R = B \oplus A$ with $A$ a left ideal. $A$ is also a right ideal because the right annihilator of $B$ in $R$ has zero intersection with $B$. Finally, consider the unitary ring $A_p = t_p(R)$. Since it is bounded as a group and since $A_p/pA_p = R/pR$ is semisimple, it is Artinian with radical $pA_p$. The remaining part is obvious. □

**Remark.** Let $A$ be an Artinian ring such that $\text{rad}(A) = pA$ for some prime $p$. Then $A$ is the product of a finite number of full matrix rings over local Artinian rings $A'$ satisfying $\text{rad}(A') = pA'$. For more information about these rings $A'$, see [6].

**References**


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