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A NOTE ON HELSON'S EXISTENCE THEOREM

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ABSTRACT. In the almost periodic context, Helson showed that there exists a unitary function in every simply invariant subspace. We provide a short proof of this result in more general setting and give some applications.

Let A be a weak-* Dirichlet algebra on probability measure space (X, φ, σ). For $1 \leqslant p < \infty$, we define the space $H^p(\sigma)$ to be $[A]_p$, the $L^p(\sigma)$-closure of A, while $H^\infty(\sigma)$ is defined to be $[A]_\infty$, the weak-* closure of A in $L^\infty(\sigma)$. A closed subspace M in $L^p(\sigma)$, $1 \leqslant p < \infty$, (a weak-* closed subspace M for $p = \infty$) is said to be invariant if $A \cdot M \subset M$. An invariant subspace M is called doubly invariant if $\bar{A} \cdot M \subset M$. We first prove the following:

THEOREM 1. *Let f be a nonnull function in $L^p(\sigma)$, $1 \leqslant p \leqslant \infty$, and let E_f be the support set of f. Then there exists a g in $[A \cdot f]_p$ such that $|g(x)| = 1$ for σ-a.e. x in E_f.*

In order to prove Theorem 1, we shall use the following well-known result which is a corollary of Szegö's theorem (see [**4**, Chapter 1], [**9**]).

LEMMA. *If f is a function in $L^p(\sigma)$, $1 \leqslant p \leqslant \infty$, such that $\log|f|$ is summable, then $f = ph$ with p unitary and h outer. The factoring is unique, up to multiplication of q and h by constant factors of modulus one.*

PROOF OF THEOREM 1. If $u = \min(1, |f|^{-1})$, then u is in $L^\infty(\sigma)$ and $\log u$ is summable. So there is an outer function h in $H^\infty(\sigma)$ such that $|h| = u$ by the lemma. Since $[A \cdot f]_p = [A \cdot hf]_p$ and $\|hf\|_\infty \leqslant 1$, we may assume that $\|f\|_\infty = 1$. Next, we put

$$F(n) = \{x;\ (n+1)^{-1} < |f(x)| \leqslant n^{-1}\}$$

for $n = 1, 2, \ldots$. Then $\sigma(E_f \setminus \bigcup_{n=1}^{\infty} F(n)) = 0$. We define

$$u_n(x) = \begin{cases} |f(x)|^{-1} & \text{for } x \text{ in } F(n), \\ 2^{-(n+1)} & \text{for } x \text{ in } X \setminus F(n). \end{cases}$$

Since $2^{-(n+1)} \leqslant u_n < n + 1$ σ-a.e., there exists an outer function h_n in $H^\infty(\sigma)$ such that $u_n = |h_n|$. Let $1 \leqslant p < \infty$. It is easy to see that $\|h_n f\|_p \leqslant \sigma(F(n)) + 2^{-(n+1)}$, so we have

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$$\sum_{n=1}^{\infty} \|h_n f\|_p \leq \sigma(E_f) + 2^{-1}.$$

Since $h_n f$ is in $[A \cdot f]_p$ ($n = 1, 2, \ldots$), it follows that $\hat{g}(x) = \sum_{n=1}^{\infty} h_n f(x)$ is in $[A \cdot f]_p$. On the other hand, we have

$$\begin{aligned} |\hat{g}(x)| &\geq |h_m f(x)| - \Big|\sum_{n \neq m} h_n f(x)\Big| \\ &> 1 - \sum_{n=1}^{\infty} 2^{-(n+1)} \\ &= 2^{-1} \quad \text{for } \sigma\text{-a.e. } x \text{ in } F(m). \end{aligned}$$

Since $|\hat{g}(x)| < 3/2$ for σ-a.e. x in X, we can find an outer function k in $H^\infty(\sigma)$ such that $k = |\hat{g}|^{-1}$ on E_f. So $g = \hat{g} \cdot k$ has the desired property. The proof is complete for $1 \leq p < \infty$. However, since $[A \cdot f]_\infty = [A \cdot f]_1 \cap L^\infty(\sigma)$ (see [4, Chapter 1, §6]), the conclusion holds for $p = \infty$.

From Theorem 1, we have this improvement of a theorem in [3].

COROLLARY. *Let M be an invariant subspace in $L^p(\sigma)$, $1 \leq p \leq \infty$. Suppose that there exists a function g in M such that the zero set of g, $Z(g)$, is σ-measure zero. Then M contains a unitary function.*

We can extend one result in [3] to the case of weak-* Dirichlet algebras. The proof is heavily dependent on Muhly's work [5].

For any subset S of $L^1(\sigma)$, we write:

$$|S| = \{|f|; f \text{ is in } S\}.$$

THEOREM 2. *Let $1 \leq p \leq \infty$. The following properties are equivalent*:

(1) *$H^\infty(\sigma)$ is a maximal weak-* closed subalgebra of $L^\infty(\sigma)$.*

(2) *For any nondoubly invariant subspace M in $L^p(\sigma)$,*

$$|M| = |H^p(\sigma)|.$$

PROOF. (1) $\Rightarrow$ (2). Let q be the conjugate index of p: $1/p + 1/q = 1$. We set

$$\tilde{M} = \{f \in L^q(\sigma); fg \text{ is in } H^1(\sigma) \text{ for any } g \text{ in } M\}.$$

Note that $\tilde{M}$ is an invariant subspace in $L^q(\sigma)$. It is easy to see that M is doubly invariant iff $M \cdot \tilde{M} = \{0\}$. Therefore there exist nonnull bounded functions h in M and k in $\tilde{M}$ such that hk is in $H^\infty(\sigma)$ (see [4, Chapter 1, §6]). Since $H^\infty(\sigma)$ is maximal, it follows that $Z(h)$ and $Z(k)$ are σ-null sets by [5, Theorem]. So M and $\tilde{M}$ contain unitary functions φ and ψ respectively by the above corollary. Since $\varphi \cdot H^p(\sigma) \subset M$ and $\psi \cdot M \subset H^1(\sigma) \cap L^p(\sigma)$, we have $|M| = |H^p(\sigma)|$.

(2) $\Rightarrow$ (1). Suppose that $H^\infty(\sigma)$ is nonmaximal. Then there exists a nonnull function f in $H^\infty(\sigma)$ such that $\sigma(Z(f)) > 0$ by [5, Theorem]. Let $M = [A \cdot f]_p$. Then M is invariant and $M \subset H^p(\sigma)$. Let $\chi_{Z(f)}$ denote the characteristic function of $Z(f)$. We can choose an outer function h such that $|h| = \chi_{Z(f)} +$

1. So $|M| \subsetneqq |H^p(\sigma)|$. Since M contains no nonconstant real-valued functions, M is not doubly invariant.

Next, we consider the condition (1) in [3]. Let X be a compact Hausdorff space upon which the real line R acts continuously as a topological transformation group. For any x in X and t in R, $x + t$ will denote the translation of x by t. Let $C[t, \infty)$ be the set of all continuous complex-valued functions f such that the spectrum of f, denoted by Sp(f), is contained in $[t, \infty)$. We write $\mathfrak{A} = C[0, \infty)$, i.e., $\mathfrak{A}$ is the algebra of all continuous analytic functions. We refer the reader to [1] for the basic facts about spectra. For any finite regular Borel measure μ, the distant future in $L^2(\mu)$ is defined to be $\bigcap_{-\infty<t<\infty} M_t(\mu)$, where $M_t(\mu)$ is the $L^2(\mu)$-closure of $C[t, \infty)$.

The following theorem is motivated by Muhly's remarks stated in [7, §5].

THEOREM 3. *If σ is an ergodic representing measure for $\mathfrak{A}$, then $\mathfrak{A}$ is a weak-* Dirichlet algebra in $L^\infty(\sigma)$. And, for any nonnegative function w in $L^p(\sigma)$, $1 \leqslant p \leqslant \infty$, the following properties are equivalent:*

(1) $\log w(x + t)$ *belongs to* $L^1(1/(1 + t^2)dt)$ *as a function of t for σ-a.e. x in X.*

(2) *There exists a nonnull function g in $H^p(\sigma)$ such that $|g| = w$.*

PROOF. Suppose that f is a function in $L^1(\sigma)$ which annihilates $\mathfrak{A} + \overline{\mathfrak{A}}$. From [1, Proposition 2′], we see that the spectrum of $f\sigma$, Sp($f\sigma$), is contained in $\{0\}$. By [6, Proposition 2.2 and Lemma 3], $|f|\sigma$ is an invariant measure. Suppose $f\sigma \neq 0$. Then, since $|f|\sigma$ is ergodic, $\mathfrak{A} + \overline{\mathfrak{A}}$ is weak-* dense in $L^\infty(|f|\sigma)$ by [6, Theorem I]. However, the Radon-Nikodym derivative $df\sigma/d|f|\sigma$ annihilates $\mathfrak{A} + \overline{\mathfrak{A}}$. This is a contradiction. Therefore $\mathfrak{A}$ is a weak-* Dirichlet algebra in $L^\infty(\sigma)$.

Now we will show the equivalence of (1) and (2). It may be assumed that σ is not a point mass, so σ is quasi-invariant (see [6, Theorem III]). Recall that $H^\infty(\sigma)$ is maximal by [6, Corollary 3.1]. Let $p = 2$. Since σ is quasi-invariant, it is easy to see that (1) holds iff the distant future of $L^2(w^2\sigma)$ is the zero subspace by [8, Theorem I]. On the other hand, $[\mathfrak{A}w]_2 = L^2(\sigma)$ iff $M_0(w^2\sigma) = L^2(w^2\sigma)$. So, since σ is ergodic quasi-invariant and not a point mass, we see that $[\mathfrak{A}w]_2$ is nondoubly invariant iff (1) holds by [8, Corollary 5.2 and Lemma 6.2]. Since $H^\infty(\sigma)$ is maximal, it follows from Theorem 2 that $|[\mathfrak{A}w]_2| = |H^2(\sigma)|$ iff $[\mathfrak{A}w]_2$ is not a doubly invariant subspace. Therefore we have the equivalence of (1) and (2) for the case $p = 2$. Let $1 \leqslant p \leqslant \infty$. Since

$$H^p(\sigma) = H^1(\sigma) \cap L^p(\sigma),$$

it suffices to consider the case $p = 1$. If $w = |g|$ where g is a nonnull function in $H^1(\sigma)$, then there is an outer function h in $H^\infty(\sigma)$ such that gh is in $H^2(\sigma)$. Since $|gh|$ has the property of (1) by the above case, it follows that (2) implies (1). The other direction is easy, so the proof is complete.

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