PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 69, Number 1, April 1978

SHAPE TRIVIALITY AND METRIC CONTRACTIONS

W. HOLSZTYŃSKI

ABSTRACT. Let (X, d) be a nonempty compact metric space such that for every $\varepsilon > 0$ there exists a map $f: X \to X$ satisfying

(i) $d(x, f(x)) < \varepsilon$ for every $x \in X$, and

(ii) $d(f(x), f(y)) < d(x, y)$ for every $x, y \in X$.

Then, as proved in this paper, the shape of X is trivial. This improves an earlier result of K. Borsuk [1], who proved that, under the same assumptions, X is acyclic.

A function $f: X \to X$ of a metric space (X, d) is called a metric contraction if

$$d(f(x), f(y)) < d(x, y) \quad \text{whenever } x, y \in X, x \neq y.$$

Answering a question posed by Nadler, Jr. [2], Borsuk proved the following:

THEOREM (BORSUK [1]). *Let (X, d) be a compact metric space satisfying the following condition*:

(*) *for every $\varepsilon > 0$ there exists a metric contraction $f: X \to X$ such that $d(f(x), x) < \varepsilon$ for every $x \in X$.*

Then X is acyclic.

In this paper we shall prove a stronger result:

THEOREM 1. *Let (X, d) be a compact metric nonempty space satisfying condition* (*). *Then the shape of X is trivial.*

First let us observe that if $f: X \to X$ is a metric contraction of a compact metric space X, then $f(A) = A$ can hold for a closed subset A of X only if A has at most one point. Thus the following theorem is more general than Theorem 1.

THEOREM 2. *Let X be a compact Hausdorff nonempty space satisfying the following condition*:

(**) *for every neighbourhood U of $\Delta_X = \{(x, x): x \in X\}$ in $X \times X$ there exists a U-shift $f: X \to X$ such that if $f(A) = A$ for a closed subset A of X then A has at most one point.*

Then X has trivial shape. ($f: X \to X$ *is called a U-shift if f is continuous and $(x, f(x)) \in U$ for every $x \in X$.*)

Received by the editors March 28, 1977.

AMS (MOS) subject classifications (1970). Primary 55D99, 54E40; Secondary 54H25.

Key words and phrases. Shape theory, trivial shape, metric contractions approximating the identity map, acyclicity, fixed point.

PROOF. We have to show that arbitrary continuous map $g: X \to P$ of X into an arbitrary finite polyhedron is homotopic to a constant map.

Let V be a neighbourhood of Δ_P in $P \times P$ such that continuous maps g_1, g_2: $X \to P$ are homotopic whenever $(g_1(x), g_2(x)) \in V$ for all $x \in X$. Since finite polyhedron P is an ANR, such V exists. Next, let $U = (g \times g)^{-1}(V)$, where $g: X \to P$ is a continuous map. Consider a map $f: X \to X$ given by (**), and the family F of all closed nonempty subsets A of X such that:

(i) $f(A) \subseteq A$,

(ii) for every neighbourhood W of A in X there exists a map $j: X \to X$ such that $j(X) \subseteq W$ and $g \circ j$ is homotopic to $g: X \to P$.

Obviously $X \in F$. It is also easy to see that the intersection of a chain of F belongs to F. Thus, by the Kuratowski-Zorn theorem, F has a minimal member A_0.

Let W be an arbitrary neighbourhood of $f(A_0)$ in X. Let $W_0 = f^{-1}(W)$. Then there exists $j: X \to X$ such that $j(X) \subseteq W_0$ and $g \circ j$ is homotopic to g. Since

$$(j(x), f \circ j(x)) \in U \quad \text{for every } x \in X,$$

hence $g \circ f \circ j$ is homotopic to $g \circ j$, i.e. $f \circ j$ is a map such that $(f \circ j)(X) \subseteq W$ and $g \circ (f \circ j)$ is homotopic to g. This means that (ii) holds for $A = f(A_0)$. Also

$$f(A) = f(f(A_0)) \subseteq f(A_0) = A.$$

Thus $f(A_0) \in F$. Since $f(A_0) \subseteq A_0$ and A_0 is a minimal member of F, hence $f(A_0) = A_0$. By (**), $A_0 = \{a\}$ is a one-point set (i.e. a is the unique fixed point of f). Let $W_1 = \{x \in X\colon (a, x) \in U\}$ and j: $X \to X$ be such that $j(X) \subseteq W_1$ and $g \circ j$ is homotopic to g. But $g \circ j: X \to P$ is homotopic also to the constant map $x \mapsto g(a)$. The theorem is proved.

REMARK. Our considerations above may be easily applied to obtain the following simple facts:

1. If X is a nonempty compact Hausdorff space and $f: X \to X$ is a continuous map, then there exists a nonempty closed subset A of X such that $f(A) = A$ and $f(B)$ is not contained in B for any proper closed subset B of A.

2. If $f: X \to X$ is a metric contraction of a compact metric space X then there exists a unique $a \in X$ such that $f(a) = a$. Furthermore, the sequence y, $f(y), f^2(y), \cdots$ is convergent to this unique fixed point a for every $y \in X$.

Indeed, let Y be the set of all limit points of the sequence y, $f(y)$, $f^2(y), \ldots$. Then $f(Y) = Y$. Thus $Y = \{a\}$ and $a = \lim_n f^n(y)$.

REFERENCES

1. K. Borsuk, *Concerning a problem due to Sam B. Nadler, Jr.*, Colloq. Math. **35** (1976), 63–65.

2. S. B. Nadler, Jr., *Some problems concerning stability of fixed points*, Colloq. Math. **27** (1973), 263–268.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WESTERN ONTARIO, LONDON, ONTARIO, CANADA

Current address: 2927 Burlington Court, Ann Arbor, Michigan 48105