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A COUNTEREXAMPLE IN THE THEORY OF FINITE GROUPS[1]

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ABSTRACT. Suppose G is a finite group and p and q are distinct odd primes. Let χ be an irreducible character of G and x and y be a p-element and a q-element of G such that $\chi(x)$, $\chi(y)$ are both irrational. In this situation it is known that G contains an element of order pq. John Thompson has asked whether y must commute with some conjugate of x. We show by example that this need not be the case.

H. F. Blichfeldt [1, Theorem 13] discovered the following remarkable theorem.

THEOREM. *Suppose G is a finite group and p and q are distinct odd prime divisors of $|G|$. Let x and y be a p-element and a q-element of G. If G has an irreducible complex character χ for which both $\chi(x)$ and $\chi(y)$ are irrational, then G contains an element of order pq.*

This result was generalized by W. Burnside [2, Chapter XVI, Theorem X]. The reader is referred to [3, Theorem (6.13)] for the statement of Burnside's theorem and a proof due to R. Brauer. According to [3, p. 41], J. Thompson has asked whether, in the situation of the theorem, y must commute with some conjugate of x. If true, this would produce a natural element of order pq. However, we shall answer Thompson's question in the negative. The author is indebted to J. Alperin of the University of Chicago for invaluable advice in this construction.

We first construct the group G. Let p be any prime such that $p - 1$ has a prime divisor $q \geqslant 5$. Let P denote the upper triangular Sylow p-group of $\mathrm{GL}(4, p)$. Thus, elements of P have diagonal entries 1, zeros below the diagonal, and arbitrary entries from $\mathrm{GF}(p)$ above the diagonal. Clearly $|P| = p^6$. Since $q|p - 1$ and $q \geqslant 5$, we can find $a, b \in \mathrm{GF}(p)$ such that a, b, and ab^{-1} are distinct primitive qth roots of 1. Let y be the diagonal matrix $\mathrm{diag}(1, a, b, 1)$, so $|y| = q$, and let G be the subgroup of $\mathrm{GL}(4, p)$ generated by y and P. Then $|G| = p^6 q$ and $P \triangle G$. A straightforward calculation shows that the center Z of P has order p and satisfies $C_P(y) = Z$. Thus, Z is also the center of G.

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Lemma 1. *Suppose φ is a faithful irreducible complex character of P. Then there is an $x \in P - Z$ with $\varphi(x)$ irrational.*

Proof. We first show that there is an $x_0 \in P - Z$ with $\varphi(x_0) \neq 0$. Indeed, if there is no such x_0, we write

$$1 = \|\varphi\|_P^2 = \frac{1}{p^6} \sum_{g \in P} |\varphi(g)|^2$$

$$= \frac{1}{p^6} \sum_{g \in Z} \varphi(1)^2 = \frac{1}{p^5} \varphi(1)^2.$$

Since $\varphi(1)$ is an integer, $\varphi(1)^2 = p^5$ is absurd. Thus, x_0 exists as claimed. Now, if $\varphi(x_0)$ is irrational, choose $x = x_0$. If $\varphi(x_0)$ is rational, choose $z \in Z^{\#}$ and set $x = x_0 z$. Since $\varphi(x) = \varphi(x_0)\varphi(z)/\varphi(1)$ and $\varphi(z)$ is irrational, we have $\varphi(x)$ irrational in this case as well.

Lemma 2. *G has a faithful irreducible complex character χ satisfying both*
(1) $\chi | P$ is faithful and irreducible,
(2) $\chi(y)$ is irrational.

Proof. We first apply Brauer's matrix duality theorem [3, Theorem (12.1)] to y acting on the character table of P. Since y centralizes Z, this theorem shows there is an irreducible character φ of P which is G-invariant and has $Z \not\subseteq \ker \varphi$. Since Z is the center of P and $|Z| = p$, we have $\ker \varphi = 1$. That is, φ is faithful. Since φ is G-invariant and $|G : P| = q$ is prime, there is an irreducible character χ_0 of G with $\chi_0 | P = \varphi$. Note that $\chi_0(y) \neq 0$, for otherwise $0 = \chi_0(y) \equiv \chi_0(1) \pmod q$ and $\chi_0(1) = \varphi(1)$ is a power of p, a contradiction. If $\chi_0(y)$ is irrational, choose $\chi = \chi_0$. If $\chi_0(y)$ is rational, let λ be a nonprincipal linear character of G/P and set $\chi = \lambda\chi_0$.

It is now clear that G is the desired counterexample. By Lemma 2(2), $\chi(y)$ is irrational and by Lemma 2(1) and Lemma 1, $\chi(x)$ is irrational for some $x \in P - Z$. However, $C_P(y) = Z$ and $x \in P - Z$, so y does not commute with any conjugate of x.

References

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