PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 69, Number 1, April 1978

AN IMPLICIT FUNCTION THEOREM WITHOUT DIFFERENTIABILITY[1]

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ABSTRACT. We combine a "global" version of the classical inverse function theorem with Schauder's fixed point theorem to investigate the existence and continuity properties of a function $(F, x) \to \eta(F, x)$ such that $\eta(F, x) = F(\eta(F, x), x)$.

Let $\mathcal{Y}$ be a Banach space, $Y, K \subset \mathcal{Y}$, Y open, K compact, X a Hausdorff space, and $F: Y \times X \to K$ a continuous function. There are two classical theorems that ensure the existence of some set $\tilde{X} \subset X$ and of an implicit function $\eta: \tilde{X} \to Y$ such that

$$\eta(x) = F(\eta(x), x) \qquad (x \in \tilde{X}),$$

namely the implicit function theorem (IFT) and Schauder's fixed point theorem. We shall combine a "global" variant of IFT with Schauder's theorem to investigate the existence and continuity of a function $(F, x) \to \eta(F, x)$ such that $\eta(F, x) = F(\eta(F, x), x)$ for $x \in X$ and for continuous F: $Y \times X \to K$ that are sufficiently "close" to some H: $Y \to K$ with a fixed point y_0.

This last problem arose in the study of the controllability at a control u_0 of the functional-integral equation

$$y(t) = \int f_0(t, \tau, \xi(y)(\tau), u(\tau))\, \mu(d\tau) \qquad (t \in T),$$

where T is a compact metric space, μ a positive nonatomic Radon measure on T, $y \in C(T, \mathbf{R}^n)$, u a control function, and ξ a "p-hereditary" [**4**, p. 203] transformation from $C(T, \mathbf{R}^n)$ to $L^\infty(\mu, \mathbf{R}^k)$. This equation, in which f_0 is Lipschitz continuous (but not necessarily differentiable) with respect to its third argument, is investigated by approximating f_0 uniformly with appropriate functions f_i which are C^1 in that argument.

Our present results are summarized in Lemma 1 and in Theorem 1 (which generalizes [**5**, Theorem 3.1, p. 20]). We write I for the identity mapping in $\mathcal{Y}$, $S(a, \alpha)$ $[S^F(a, \alpha)]$ for the open [closed] ball in $\mathcal{Y}$ of center a and radius α,

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Received by the editors June 7, 1977.

AMS (MOS) subject classifications (1970). Primary 26A57, 47H15; Secondary 26A96, 47H10, 54C65.

Key words and phrases. Implicit functions, nondifferentiable functions, fixed points, Banach spaces, measurable and continuous selections.

[1]Partially supported by Grant MCS 76-06756 of the National Science Foundation.

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$d[A,y]$ for the distance between a point y and a set A, $\overset{\Delta}{=}$ for "equal by definition", and

$$S(A,\alpha) \overset{\Delta}{=} \{y \in \mathcal{Y} \,|\, d[A,y] < \alpha\}.$$

THEOREM 1. *Let $\mathcal{Y}$ be a Banach space, Y an open subset of $\mathcal{Y}$, K a compact subset of $\mathcal{Y}$, and X a Hausdorff space. Suppose that $0 < \alpha, c < \infty$, $S^F(y_0, \alpha) \subset Y$, G_0: $Y \to K$ is C^1, and*

$$\left|\left[I - G_0'(y)\right]^{-1}\right| \leqslant c \qquad (y \in Y).$$

Let $\tilde{K} \overset{\Delta}{=} K \cap S^F(y_0, \alpha)$, and let $\mathcal{F}$ be the metric space of all continuous functions F: $Y \times X \to K$ such that

$$|F(y,x) - G_0(y) + G_0(y_0) - y_0| \leqslant \alpha/c \qquad \left[x \in X, y \in S^F(y_0, \alpha)\right],$$

with two elements of $\mathcal{F}$ identified if they coincide on $\tilde{K} \times X$, and with the metric

$$\rho(F, F_1) \overset{\Delta}{=} \sup\left\{|F(y,x) - F_1(y,x)| \,\big|\, (y,x) \in \tilde{K} \times X\right\}.$$

Then

(I) *For each $(F, x) \in \mathcal{F} \times X$, the equation $y = F(y,x)$ has a solution $y \in \tilde{K}$;*

(II) *There exists a Borel measurable function η: $\mathcal{F} \times X \to \tilde{K}$ such that*

$$\eta(F,x) = F(\eta(F,x), x) \qquad (F \in \mathcal{F}, x \in X);$$

(III) *If $\bar{F} \in \mathcal{F}$, $\bar{x} \in X$ and if $\Phi(F,x) \overset{\Delta}{=} \{y \in \tilde{K} \,|\, y = F(y,x)\}$ is a singleton at $(\bar{F}, \bar{x})$ then every selection $\tilde{\eta}$ of the set-valued mapping Φ is continuous at $(\bar{F}, \bar{x})$;*

(IV) *If X is compact, $F \in \mathcal{F}$, and $\Phi(F,x)$ is the singleton $\{\eta(F,x)\}$ for each $x \in X$, then $\eta(F, \cdot\,)$ is continuous and*

$$\lim_{F_1 \to F} \tilde{\eta}(F_1, x) = \eta(F, x) \quad \textit{uniformly on } X$$

for every selection $\tilde{\eta}$ of Φ.

REMARK (ADDED IN PROOF). The following, more general, proposition can be proved exactly as statement (I): Let G: $Y \to \mathcal{Y}$ be C^1, $|G'(y)^{-1}| \leqslant c$ $(y \in Y)$, $G(y_0) = 0$, and let Γ: $Y \to K$ be continuous and such that $|\Gamma(y)| \leqslant \beta < \alpha/c$ $(y \in Y)$. Then the equation $G(y) + \Gamma(y) = z$ has a solution $y \in S^F(y_0, \alpha)$ for every $z \in \mathcal{Y}$ with $|z| \leqslant \alpha/c - \beta$.

In order to prove Theorem 1 we shall first require a "global" version of the classical inverse function theorem.

LEMMA 1. *Let Y be an open subset of a Banach space $\mathcal{Y}$, $0 < \alpha, c < \infty$, $S^F(y_0, \alpha) \subset Y$ and G: $Y \to \mathcal{Y}$ a C^1 function such that*

$$G(y_0) = 0, \quad \left|G'(y)^{-1}\right| \leqslant c \qquad (y \in Y).$$

Then there exists a unique C^1 function u: $S^F(0, \alpha/c) \to S^F(y_0, \alpha)$ such that

$$G(u(x)) = x \quad (|x| \leqslant \alpha/c), \quad u(0) = y_0.$$

PROOF. We first recall that, by the classical inverse and implicit function theorems (as stated, e.g. in [2, pp. 265, 268]), for every point $\eta_0 \in Y$ and a corresponding $x_0 \stackrel{\Delta}{=} G(\eta_0)$, there exist $\varepsilon_0 > 0$ and a unique C^1 function v_0: $S(x_0, \varepsilon_0) \to Y$ such that

$$G(v_0(x)) = x \quad (|x - x_0| < \varepsilon_0), \quad v_0(x_0) = \eta_0.$$

Furthermore, if A is a connected open subset of $\mathcal{Y}$, $x_0 \in A$, and u_1: $A \to Y$ and u_2: $A \to Y$ are two continuous functions such that

$$G(u_i(x)) = x \quad (i = 1, 2, x \in A), \quad u_1(x_0) = u_2(x_0)$$

then $u_1 = u_2$, u_1 is C^1 and $u_1'(x) = G'(u_1(x))^{-1}$, $(x \in A)$. We shall henceforth refer to the above assertions as IFT.

Let $a \in \mathcal{Y}$ and $|a| = 1$. We shall denote by $\mathfrak{B}$ the collection of all points $\beta \in [0, \alpha/c]$ such that there exist $\varepsilon_\beta > 0$, a corresponding open convex set $U_\beta =^\Delta S([0, \beta]a, \varepsilon_\beta)$ and a C^1 function v_β: $U_\beta \to Y$ satisfying

$$G(v_\beta(x)) = x \quad (x \in U_\beta), \quad v_\beta(0) = y_0.$$

It follows from IFT that $0 \in \mathfrak{B}$ and it is clear that $\mathfrak{B}$ is a relatively open subinterval of $[0, \alpha/c]$.

Now let $\bar{\beta} \stackrel{\Delta}{=} \sup \mathfrak{B}$ and $U \stackrel{\Delta}{=} \bigcup_{\beta \in \mathfrak{B}} U_\beta$. Then U is an open and connected neighborhood of $\mathfrak{B}a$ and, by IFT, $v_\beta(x) = v_\gamma(x)$ if $\beta, \gamma \in \mathfrak{B}$ and $x \in U_\beta \cap U_\gamma$. We may therefore define a unique C^1 function v: $U \to Y$ by

$$v(x) = v_\beta(x) \qquad (\beta \in \mathfrak{B}, x \in U_\beta).$$

We have $|v'(x)| = |G'(v(x))^{-1}| \leqslant c$ and therefore

$$|v(\beta a) - v(0)| = |v(\beta a) - y_0| \leqslant c\beta \leqslant \alpha \quad (\beta \in \mathfrak{B}).$$

Thus $w \stackrel{\Delta}{=} \lim v(\beta a)$, $(\beta \to \bar{\beta}, \beta \in \mathfrak{B})$ exists, $w \in S^F(y_0, \alpha) \subset Y$ and $G(w) = \bar{\beta}a$. Again, by IFT, there exists $\bar{\varepsilon} > 0$ and a unique C^1 function $\bar{v}$: $S(\bar{\beta}a, \bar{\varepsilon}) \to Y$ satisfying

$$G(\bar{v}(x)) = x \quad (|x - \bar{\beta}a| < \bar{\varepsilon}), \quad \bar{v}(\bar{\beta}a) = w,$$

$$\bar{v}(x) = v(x) \qquad \left(\beta \in \mathfrak{B}, x \in U_\beta \cap S(\bar{\beta}a, \bar{\varepsilon})\right).$$

Since $\beta' \stackrel{\Delta}{=} \bar{\beta} - \frac{1}{2}\bar{\varepsilon} \in \mathfrak{B}$, the function v is defined on $A_{\beta'} \stackrel{\Delta}{=} S([0, \beta']a, \varepsilon')$, where $\varepsilon' \stackrel{\Delta}{=} \mathrm{Min}(\frac{1}{2}\bar{\varepsilon}, \varepsilon_{\beta'}) > 0$. We may therefore define a C^1 function u^a by

$$U^a \stackrel{\Delta}{=} S\left(\left[0, \bar{\beta}\right]a, \varepsilon'\right), \quad u^a(x) = v(x) \quad (x \in A_{\beta'}),$$

$$u^a(x) = \bar{v}(x) \quad (x \in U^a \sim A_{\beta'}),$$

which shows that $\bar{\beta} = \sup \mathfrak{B} \in \mathfrak{B}$. Thus $\mathfrak{B}$ is a nonempty, open and closed subset of $[0, \alpha/c]$; hence $\mathfrak{B} = [0, \alpha/c]$.

We now conclude that for every $a \in \mathcal{Y}$ with $|a| = 1$ there exist an open connected neighborhood U^a of $[0, \alpha/c]a$ and a unique continuous function v^a: $U^a \to Y$ such that

$$G(v^a(x)) = x \quad (x \in U^a), \quad v^a(0) = y_0.$$

Furthermore, by IFT, $v^a(x) = v^b(x)$ if $x \in U^a \cap U^b$. We may therefore define a unique continuous $u: S^F(0, \alpha/c) \to Y$ by

$$u(x) = v^a(x) \qquad (a \in \mathcal{Y}, |a| = 1, x \in U^a),$$

and it follows from IFT that u is C^1 and $|u(x) - y_0| \leqslant c|x| \leqslant \alpha$ for all $x \in S^F(0, \alpha/c)$. Q.E.D.

PROOF OF THEOREM 1. Let $H(y) \triangleq G_0(y) + y_0 - G_0(y_0)$. Then, by Lemma 1, there exists a unique C^1 function $u: S^F(0, \alpha/c) \to S^F(y_0, \alpha)$ such that

$$(1) \qquad (I - H)(u(v)) = v \quad (|v| \leqslant \alpha/c), \quad u(0) = y_0.$$

If $F \in \mathfrak{F}$ and $x \in X$ then $|F(y, x) - H(y)| \leqslant \alpha/c$ $[y \in S^F(y_0, \alpha)]$. Thus $y \to u(F(y, x) - H(y))$ is a continuous mapping of $S^F(y_0, \alpha)$ into itself. Furthermore, since $F(y, x) - H(y) \in K - K + G_0(y_0) - y_0$, this mapping carries $S^F(y_0, \alpha)$ into a compact set. Therefore, by Schauder's fixed point theorem, this mapping has a fixed point $\bar{\eta}$. In view of (1), we have

$$\bar{\eta} = u\big(F(\bar{\eta}, x) - H(\bar{\eta})\big) = H(\bar{\eta}) + F(\bar{\eta}, x) - H(\bar{\eta}) = F(\bar{\eta}, x).$$

This proves statement (I).

Now let

$$\Phi(F, x) \triangleq \{y \in \tilde{K} | y = F(y, x)\} \qquad (F \in \mathfrak{F}, x \in X),$$

$$\text{Graph }(\Phi) \triangleq \{((F, x), y) | y \in \Phi(F, x)\}.$$

Then, by statement (I), the set $\Phi(F, x)$ is nonempty for all $(F, x) \in \mathfrak{F} \times X$, and it is easy to see that each $\Phi(F, x)$ is closed in the compact space $\tilde{K}$ and Graph (Φ) is closed in $(\mathfrak{F} \times X) \times \tilde{K}$. It follows that

(i) the set-valued mapping Φ is Borel measurable, that is, for each closed $C \subset \tilde{K}$, the set $\{(F, x) | \Phi(F, x) \cap C \neq \varnothing\}$ is Borel measurable (and, in fact, closed); and

(ii) by a theorem of Berge [**1**, Corollary to Theorem 7, p. 112], for every $(\bar{F}, \bar{x}) \in \mathfrak{F} \times X$ and every open subset U of the space $\tilde{K}$, with $\Phi(\bar{F}, \bar{x}) \subset U$, there exists a neighborhood V of $(\bar{F}, \bar{x})$ in $\mathfrak{F} \times X$ such that $\Phi(F, x) \subset U$ for all $(F, x) \in V$.

By a known measurable selection theorem [**3**, Theorem 4.1, p. 867], it follows from (i) that Φ has a Borel measurable selection $(F, x) \to \eta(F, x)$, which proves statement (II). Statement (III) follows directly from (ii), and (IV) follows from (III). Q.E.D.

ACKNOWLEDGEMENT. I wish to thank Professor Togo Nishiura for suggesting improvements in Theorem 1 and in the proof of statements (II)–(IV).

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