AN IMPLICIT FUNCTION THEOREM
WITHOUT DIFFERENTIABILITY

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Abstract. We combine a “global” version of the classical inverse function
theorem with Schauder’s fixed point theorem to investigate the existence
and continuity properties of a function \((F, x) \rightarrow \eta(F, x)\) such that \(\eta(F, x) =
F(\eta(F, x), x)\).

Let \(\mathfrak{g}\) be a Banach space, \(Y, K \subset \mathfrak{g}\), \(Y\) open, \(K\) compact, \(X\) a Hausdorff
space, and \(F: Y \times X \rightarrow K\) a continuous function. There are two classical
theorems that ensure the existence of some set \(\tilde{X} \subset X\) and of an implicit
function \(\eta: \tilde{X} \rightarrow Y\) such that
\[
\eta(x) = F(\eta(x), x) \quad (x \in \tilde{X}),
\]

namely the implicit function theorem (IFT) and Schauder’s fixed point
theorem. We shall combine a “global” variant of IFT with Schauder’s
theorem to investigate the existence and continuity of a function \((F, x) \rightarrow
\eta(F, x)\) such that \(\eta(F, x) = F(\eta(F, x), x)\) for \(x \in X\) and for continuous \(F:
Y \times X \rightarrow K\) that are sufficiently “close” to some \(H: Y \rightarrow K\) with a fixed
point \(y_0\).

This last problem arose in the study of the controllability at a control \(u_0\) of
the functional-integral equation
\[
y(t) = \int f_0(t, \tau, \xi(y)(\tau), u(\tau)) \mu(d\tau) \quad (t \in T),
\]

where \(T\) is a compact metric space, \(\mu\) a positive nonatomic Radon measure on
\(T\), \(y \in C(T, \mathbb{R}^n)\), \(u\) a control function, and \(\xi\) a “\(p\)-hereditary” \([4, \text{p. 203}]\)
transformation from \(C(T, \mathbb{R}^n)\) to \(L^\infty(\mu, \mathbb{R}^k)\). This equation, in which \(f_0\) is
Lipschitz continuous (but not necessarily differentiable) with respect to its
third argument, is investigated by approximating \(f_0\) uniformly with
appropriate functions \(f\) which are \(C^1\) in that argument.

Our present results are summarized in Lemma 1 and in Theorem 1 (which
generalizes \([5, \text{Theorem 3.1, p. 20}]\)). We write \(I\) for the identity mapping in \(\mathfrak{g}\),
\(S(a, a) \ [S^c(a, a)]\) for the open [closed] ball in \(\mathfrak{g}\) of center \(a\) and radius \(a\),
For the distance between a point \( y \) and a set \( A \), \( \Delta \) for "equal by definition", and
\[
S(A, \alpha) \triangleq \{ y \in \mathbb{B} | d[A, y] < \alpha \}.
\]

**Theorem 1.** Let \( \mathbb{B} \) be a Banach space, \( Y \) an open subset of \( \mathbb{B} \), \( K \) a compact subset of \( \mathbb{B} \), and \( X \) a Hausdorff space. Suppose that \( 0 < \alpha, c < \infty \), \( S^F(y_0, \alpha) \subset Y \), \( G_0: Y \to K \) is \( C^1 \), and
\[
\left| \left[ I - G_0(y) \right]^{-1} \right| < c \quad (y \in Y).
\]
Let \( \tilde{K} \triangleq K \cap S^F(y_0, \alpha) \), and let \( \mathcal{T} \) be the metric space of all continuous functions \( F: Y \times X \to K \) such that
\[
|F(y, x) - G_0(y) + G_0(y_0) - y_0| < \frac{\alpha}{c} \quad [x \in X, y \in S^F(y_0, \alpha)],
\]
with two elements of \( \mathcal{T} \) identified if they coincide on \( \tilde{K} \times X \), and with the metric
\[
\rho(F, F_1) \triangleq \sup \{ |F(y, x) - F_1(y, x)| \mid (y, x) \in \tilde{K} \times X \}.
\]
Then
(I) For each \( (F, x) \in \mathcal{T} \times X \), the equation \( y = F(y, x) \) has a solution \( y \in \tilde{K} \);

(II) There exists a Borel measurable function \( \eta: \mathcal{T} \times X \to \tilde{K} \) such that
\[
\eta(F, x) = F(\eta(F, x), x) \quad (F \in \mathcal{T}, x \in X);
\]

(III) If \( F \in \mathcal{T} \), \( x \in X \) and if \( \Phi(F, x) \triangleq \{ y \in \tilde{K} | y = F(y, x) \} \) is a singleton at \( (F, x) \) then every selection \( \bar{\eta} \) of the set-valued mapping \( \Phi \) is continuous at \( (F, \bar{x}) \);

(IV) If \( X \) is compact, \( F \in \mathcal{T} \), and \( \Psi(F, x) \) is the singleton \( \{ \eta(F, x) \} \) for each \( x \in X \), then \( \eta(F, \cdot) \) is continuous and
\[
\lim_{F_i \to F} \eta(F_i, x) = \eta(F, x) \quad \text{uniformly on } X
\]
for every selection \( \bar{\eta} \) of \( \Phi \).

**Remark (added in proof).** The following, more general, proposition can be proved exactly as statement (I): Let \( G: Y \to \mathbb{B} \) be \( C^1 \), \( |G'(y)| < c \) \( (y \in Y) \), \( G(y_0) = 0 \), and let \( \Gamma: Y \to K \) be continuous and such that \( |\Gamma(y)| < \beta < \alpha/c \) \( (y \in Y) \). Then the equation \( G(y) + \Gamma(y) = z \) has a solution \( y \in S^F(y_0, \alpha) \) for every \( z \in \mathcal{B} \) with \( |z| < \alpha/c - \beta \).

In order to prove Theorem 1 we shall first require a "global" version of the classical inverse function theorem.

**Lemma 1.** Let \( Y \) be an open subset of a Banach space \( \mathbb{B} \), \( 0 < \alpha, c < \infty \), \( S^F(y_0, \alpha) \subset Y \) and \( G: Y \to \mathcal{B} \) a \( C^1 \) function such that
\[
G(y_0) = 0, \quad |G'(y)| < c \quad (y \in Y).
\]
Then there exists a unique \( C^1 \) function \( u: S^F(0, \alpha/c) \to S^F(y_0, \alpha) \) such that
\[
G(u(x)) = x \quad (|x| < \alpha/c), \quad u(0) = y_0.
\]

**Proof.** We first recall that, by the classical inverse and implicit function
theorems (as stated, e.g. in [2, pp. 265, 268]), for every point \( \eta_0 \in Y \) and a corresponding \( x_0 \triangleq G(\eta_0) \), there exist \( \varepsilon_0 > 0 \) and a unique \( C^1 \) function \( v_0: S(x_0, \varepsilon_0) \to Y \) such that
\[
G(v_0(x)) = x \quad (|x - x_0| < \varepsilon_0), \quad v_0(x_0) = \eta_0.
\]
Furthermore, if \( A \) is a connected open subset of \( \mathbb{G} \), \( x_0 \in A \), and \( u_1: A \to Y \) and \( u_2: A \to Y \) are two continuous functions such that
\[
G(u_i(x)) = x \quad (i = 1, 2, x \in A), \quad u_1(x_0) = u_2(x_0)
\]
then \( u_1 = u_2 \), \( u_1 \) is \( C^1 \) and \( G'(u_1(x))^{-1} \), \( (x \in A) \). We shall henceforth refer to the above assertions as IFT.

Let \( a \in \mathbb{G} \) and \( |a| = 1 \). We shall denote by \( B \) the collection of all points \( \beta \in [0, \alpha/c] \) such that there exist \( \varepsilon_\beta > 0 \), a corresponding open convex set \( U_\beta = \Delta S([0, \beta]a, \varepsilon_\beta) \) and a \( C^1 \) function \( v_\beta: U_\beta \to Y \) satisfying
\[
G(v_\beta(x)) = x \quad (x \in U_\beta), \quad v_\beta(0) = y_0.
\]
It follows from IFT that \( 0 \in B \) and it is clear that \( B \) is a relatively open subinterval of \([0, \alpha/c]\).

Now let \( \bar{\beta} = \sup B \) and \( U = \Delta \bigcup_{\beta \in B} U_\beta \). Then \( U \) is an open and connected neighborhood of \( \mathbb{G}a \) and, by IFT, \( v_\beta(x) = v_\gamma(x) \) if \( \beta, \gamma \in B \) and \( x \in U_\beta \cap U_\gamma \). We may therefore define a unique \( C^1 \) function \( v: U \to Y \) by
\[
v(x) = v_\beta(x) \quad (\beta \in B, x \in U_\beta).
\]
We have \( |v'(x)| = |G'(v(x))^{-1}| < c \) and therefore
\[
|v(\beta a) - v(0)| = |v(\beta a) - y_0| < c\beta < \alpha \quad (\beta \in B).
\]
Thus \( w = \lim v(\beta a), \ (\beta \to \bar{\beta}, \beta \in B) \) exists, \( w \in SF(y_0, \alpha) \subset Y \) and \( G(w) = \bar{\beta}a \). Again, by IFT, there exists \( \varepsilon > 0 \) and a unique \( C^1 \) function \( \bar{v}: S(\bar{\beta}a, \varepsilon) \to Y \) satisfying
\[
G(\bar{v}(x)) = x \quad (|x - \bar{\beta}a| < \varepsilon), \quad \bar{v}(\bar{\beta}a) = w.
\]
Since \( \beta' = \bar{\beta} - \frac{1}{2}\varepsilon \in B \), the function \( v \) is defined on \( A_{\beta'} = \Delta S([0, \beta']a, \varepsilon') \), where \( \varepsilon' = \min(\frac{1}{2}\varepsilon, \varepsilon_\beta) > 0 \). We may therefore define a \( C^1 \) function \( u^a \) by
\[
U^a = \Delta S([0, \bar{\beta}]a, \varepsilon'), \quad u^a(x) = v(x) \quad (x \in A_{\beta'}),
\]

\[
u^a(x) = \bar{v}(x) \quad (x \in U^a \sim A_{\beta'}),
\]
which shows that \( \bar{\beta} = \sup B \in B \). Thus \( B \) is a nonempty, open and closed subset of \([0, \alpha/c]\); hence \( B = [0, \alpha/c] \).

We now conclude that for every \( a \in \mathbb{G} \) with \( |a| = 1 \) there exist an open connected neighborhood \( U^a \) of \([0, \alpha/c]a \) and a unique continuous function \( v^a: U^a \to Y \) such that
\[
G(v^a(x)) = x \quad (x \in U^a), \quad v^a(0) = y_0.
\]
Furthermore, by IFT, \( v^a(x) = v^b(x) \) if \( x \in U^a \cap U^b \). We may therefore
define a unique continuous \( u: S^F(0, \alpha/c) \to Y \) by

\[
u(x) = v^\alpha(x) \quad (a \in \mathcal{O}, |a| = 1, x \in U^\alpha),
\]

and it follows from IFT that \( u \) is \( C^1 \) and \( |u(x) - y_0| < c|x| < \alpha \) for all \( x \in S^F(0, \alpha/c) \). Q.E.D.

**Proof of Theorem 1.** Let \( H(y) = G_0(y) + y_0 - G_0(y_0) \). Then, by Lemma 1, there exists a unique \( C^1 \) function \( u: S^F(0, \alpha/c) \to S^F(y_0, \alpha) \) such that

\[
(1) \quad (I - H)(u(v)) = v \quad (|v| < \alpha/c), \quad u(0) = y_0.
\]

If \( F \in \mathcal{G} \) and \( x \in X \) then \( |F(y, x) - H(y)| < \alpha/c \) \( [y \in S^F(y_0, \alpha)] \). Thus \( y \to u(F(y, x) - H(y)) \) is a continuous mapping of \( S^F(y_0, \alpha) \) into itself. Furthermore, since \( F(y, x) - H(y) \in K - K + G_0(y_0) - y_0 \), this mapping carries \( S^F(y_0, \alpha) \) into a compact set. Therefore, by Schauder's fixed point theorem, this mapping has a fixed point \( \eta \). In view of (1), we have

\[
\eta = u(F(\eta, x) - H(\eta)) = H(\eta) + F(\eta, x) - H(\eta) = F(\eta, x).
\]

This proves statement (I).

Now let

\[
\Phi(F, x) = \{ y \in \bar{K} | y = F(y, x) \} \quad (F \in \mathcal{G}, x \in X),
\]

\[
\text{Graph} (\Phi) = \{ ((F, x), y) | y \in \Phi(F, x) \}.
\]

Then, by statement (I), the set \( \Phi(F, x) \) is nonempty for all \( (F, x) \in \mathcal{G} \times X \), and it is easy to see that each \( \Phi(F, x) \) is closed in the compact space \( \bar{K} \) and \( \text{Graph} (\Phi) \) is closed in \( (\mathcal{G} \times X) \times \bar{K} \). It follows that

(i) the set-valued mapping \( \Phi \) is Borel measurable, that is, for each closed \( C \subset \bar{K} \), the set \( \{ (F, x) | \Phi(F, x) \cap C \neq \emptyset \} \) is Borel measurable (and, in fact, closed); and

(ii) by a theorem of Berge [1, Corollary to Theorem 7, p. 112], for every \( (\bar{F}, \bar{x}) \in \mathcal{G} \times X \) and every open subset \( U \) of the space \( \bar{K} \), with \( \Phi(\bar{F}, \bar{x}) \subset U \), there exists a neighborhood \( V \) of \( (\bar{F}, \bar{x}) \) in \( \mathcal{G} \times X \) such that \( \Phi(F, x) \subset U \) for all \( (F, x) \in V \).

By a known measurable selection theorem [3, Theorem 4.1, p. 867], it follows from (i) that \( \Phi \) has a Borel measurable selection \( (F, x) \to \eta(F, x) \), which proves statement (II). Statement (III) follows directly from (ii), and (IV) follows from (III). Q.E.D.

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**References**


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