PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 69, Number 1, April 1978

ON STRONGLY EXPOSING FUNCTIONALS

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ABSTRACT. Let K be a weakly compact convex set. The recent result of Lau that its strongly exposing functionals form a dense G_δ is here proved by a modification of Lindenstrauss's proof that K is the closed convex hull of its strongly exposed points.

Recently Lau [3] has examined conditions which guarantee that the strongly exposing functionals of a closed convex set form a dense G_δ set in the dual norm. The central theorem was improved by Lau in [4] to show that any weakly compact convex set has this property. This proof relies heavily on the renorming theorem of Trojanski [7] and on results on the existence of farthest points [3]. Bourgain [1] has also provided a proof, modeled on Phelps [6] characterization of the Radon-Nikodym property, which is, as he says, "purely geometric" in nature. It is the purpose of this note to indicate that the original proof method of Lindenstrauss [5] can be easily adapted to show the desired result rather than just showing that a weakly compact convex set is the closed convex hull of its strongly exposed points.

Let X be a normed linear space, X^* its dual space. Suppose K is a closed, bounded convex subset of X: a point $x \in K$ is a *strongly exposed* point of K if there is a linear functional $f \in X^*$ such that $f(x) > f(y)$ if $y \in K \sim \{x\}$ and such that whenever $\{f(x_n)\}$ converges to $f(x)$ and $\{x_n\}$ is in K we have, in fact, $\{x_n\}$ converging to x (in norm). The functional f is said to be *strongly exposing*. Let $K^\frown$ denote the strongly exposing functionals of K.

We quote the following results which will be used in the sequel.

LEMMA 1 [6]. *Let X be a normed linear space. For any $\varepsilon > 0$, suppose there exist $f, g \in X^*$ with $\|f\| = \|g\| = 1$ and suppose $f(x) \leqslant \varepsilon/2$ whenever $g(x) = 0$ and $\|x\| \leqslant 1$. Then either $\|f - g\| \leqslant \varepsilon$ or $\|f + g\| \leqslant \varepsilon$.*

LEMMA 2 [5]. *Let X, Y be Banach spaces and let T be a bounded linear operator between X and Y. For any $\varepsilon > 0$ and any weakly compact convex set K, there is another bounded linear operator S such that $\|T - S\| < \varepsilon$ and such that* $\sup\{\|Sk\|: k \in K\} = \|Sk_0\|$ *for some $k_0 \in K$.*

THEOREM. *If K is a weakly compact convex set in a normed space then $K^\frown$ is a dense G_δ set in X^*.*

Received by the editors March 8, 1977.
AMS (MOS) subject classifications (1970). Primary 46B99; Secondary 52A05.
Key words and phrases. Strongly exposing functionals, weakly compact convex sets.

PROOF. The proof is an adaptation of that given in [2]. We may assume, as in [2], that X is the closed linear span of K and so, being weakly compactly generated, has a locally uniformly convex norm [2], [7]. It is easily shown that $K^{\frown}$ is always a G_δ subset [1], [3]. We need only show now that $K^{\frown}$ is norm dense in X^*.

Let $g \in X^*$ with $\|g\| = 1$ and let $0 < \varepsilon < 1/3$ be given. Since $K^{\frown} = (rK + x)^{\frown}$ for any $x \in X$, $r \in R$ we may assume that $\|k\| \leqslant 1$ and $g(k) > \varepsilon$ for all $k \in K$. Now let

$$C = \{x \in X: \|x\| \leqslant 2/\varepsilon, g(x) = 0\}; \qquad D = \overline{\text{conv}}\{K \cup C\}.$$

Select $f \in X^*$, $k_0 \in K$, and $0 < \delta < 1$, such that $\sup\{|f(k)|: k \in K\} \geqslant 1 = f(k_0)$ while $\sup\{f(c): c \in C\} \leqslant 1 - \delta$. This can be done since C is a symmetric set disjoint from K. Let Y be the direct sum of X and R endowed with the l_2 sum of the norms on X and R as its norm. Define $T: X \to Y$ by $Tx = (x, Mf(x))$ where M is chosen so that

$$M^2(1-\delta)^2 + 4/\varepsilon^2 \leqslant (M-1)^2.$$

($M \geqslant \max(2/\delta, 4 - \varepsilon^2/\varepsilon^2(\delta - \delta^2))$ suffices.)

Then $\|Tk_0\| \geqslant M$ and if $c \in C$

$$\|Tc\|^2 \leqslant \|c\|^2 + M^2|f(c)|^2 \leqslant (M-1)^2.$$

Since K is weakly compact we can find an operator S and $k_1 \in K$ such that

$$\|S - T\| \leqslant \varepsilon^2 \quad \text{and} \quad \|Sk_1\| = \sup\{\|Sk\|: k \in K\}.$$

(Note that we may assume that S is an isomorphism into Y since T is.) Thus

$$\|Sk_1\| \geqslant \sup\{\|Tk\|: k \in K\} - \varepsilon^2 > M - 1/3,$$

while for $c \in C$

$$\|Sc\| \leqslant \sup\{\|(T-S)c\|: c \in C\} + (M-1)$$
$$\leqslant 2\varepsilon^2/\varepsilon + M - 1 \leqslant M - 1/3.$$

It follows that S achieves its bounds at k_1 not only on K but also on D. As in [2], pick $e \in Y^*$ of unit norm with $e(Sk_1) = \|Sk_1\| \geqslant e(Sd)$ for all $d \in D$. Let $h = eS$. Then $h \in X^*$ and strongly exposes k_1 on D. For, let $x_n \in D$ and suppose that $h(x_n) \to h(k_1)$. Now

$$h(x_n + k_1) = e(Sx_n) + e(Sk_1) \to 2\|Sk_1\|.$$

$\|e\| = 1$ so that $\|Sx_n + Sk_1\| \to 2\|Sk_1\|$. $\|Sx_n\| \leqslant \|Sk_1\|$, and this implies that $Sx_n \to Sk_1$ because the norm is locally uniformly convex. Since S is an isomorphism $\|x_n - k_1\| \to 0$ and h strongly exposes k_1 on D. Also, if $g(x) = 0$ and $\|x\| \leqslant 1$, $\pm(2/\varepsilon)x \in D$ and $|h(2x/\varepsilon)| \leqslant h(k_1)$. Thus if $g_1 = h/\|h\|$ we have: $g(x) = 0$ and $\|x\| \leqslant 1$ implies $|g_1(x)| \leqslant \varepsilon/2$. By Lemma 1 either $\|g - g_1\| \leqslant \varepsilon$ or $\|g + g_1\| \leqslant \varepsilon$. However,

$$g(k_1) + g_1(k_1) \geqslant g(k_1) + g_1(0) > \varepsilon$$

since g_1 exposes D at k_1 and $0 \in D$. Since $\|k_1\| \leqslant 1$, $\|g + g_1\| > \varepsilon$. Thus

$\|g - g_1\| < \varepsilon$ and we have constructed a strongly exposing functional for $K(D)$ within ε of g. Thus $K^{\frown}$ is dense in X^*. The fact that K is the closed convex hull of its strongly exposed points now follows from a standard separation argument [2].

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