VECTOR-VALUED CONTINUOUS FUNCTIONS
WITH STRICT TOPOLOGIES
AND ANGELIC TOPOLOGICAL SPACES

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ABSTRACT. It is proved that if $X$ is a metric space, $E$ a Banach space
containing a $\sigma$-weakly-compact dense subset, then the space $(M_r(X, E'),
\sigma(M_r(X, E'), C_b(X, E)))$ is angelic, $C_b(X, E)$ being all bounded continuous
functions from $X$ into $E$ and $M_r(X, E')$ the dual of $C_b(X, E)$ with the strict
topology $\beta$.

A Hausdorff topological space $Y$ is called angelic if (i) every relatively
countably compact subset of $Y$ is relatively compact, and (ii) for any point $x$
in the closure of a relatively compact subset $A$ of $Y$, there exists a sequence,
in $A$, converging to $x$ [4, p. 534]. In this paper $X$ will stand for a completely
regular Hausdorff space, $E$ a Banach space over $K$, the field of real or
complex numbers, $C(X, E)$ ($C(X)$) all $E$-valued ($K$-valued) continuous
functions on $X$, and $C_b(X, E)$ ($C_b(X)$) all bounded $E$-valued ($K$-valued)
continuous functions on $X$. We shall use the notations of [5] for locally
convex spaces. Also the notations and results from [2] will be used. The
topologies $\beta, \beta_0, \beta_1, \beta_e$ (also denoted by $\beta_{\infty}$) are defined on $C_b(X)$ in [9], [7],
[8]; these topologies are defined for $K = R$, the reals, but naturally extend to
$K = C$, the complex field. $\beta, \beta_0, \beta_1, \beta_{\infty}$ are defined on $C_b(X, E)$ in [1], [2]. If
$\mathfrak{C}^\infty = \mathfrak{C}^\infty(X, E) = \{H \subset C_b(X, E): H$ pointwise equicontinuous and
uniformly bounded}, the topology $\beta_{\infty}$ is the finest locally convex topology on
$C_b(X, E)$ agreeing with pointwise topology on each $H \in \mathfrak{C}^\infty$. It is well
known that $M_r(X, E') = (C_b(X, E), \beta)'$, $M_r(X, E') = (C_b(X, E), \beta_0)'$ and
$M_r(X, E') = (C_b(X, E), \beta_e)'$ (see [2]). If $X$ is a metric, $M_{\infty}(X, E') =
M_r(X, E')$ and $\beta$ and $\beta_{\infty}$ are both Mackey [2]; from this it follows that
$\beta = \beta_{\infty}$ in this case.

If $X$ is a metric space it is proved in [5, Theorem 5.3] that
$(M_{\infty}(X), \sigma(M_{\infty}(X), C_b(X)))$ is angelic. We will extend this result to
$M_{\infty}(X, E')$ for some special Banach spaces $E$.

THEOREM 1. $(F', \tau(F', F))$ is complete, $\tau$ denoting the Mackey topology of the
duality $\langle F, F' \rangle$, where $F = C_b(X, E)$ and $F' = M_{\infty}(X, E')$.

PROOF. By Grothendieck completeness theorem [5, Theorem 6.2, p. 148], it
is enough to prove that any linear form $\mu$ on $F$ which is continuous on every

Received by the editors May 6, 1977 and, in revised form, August 8, 1977.

absolutely convex compact subset \((F, \sigma(F, F'))\) with topology induced by \(\beta_\infty\), belongs to \(F'\). Considering \(\mu: (C_b(X, E), \|\cdot\|) \to K\), we first prove that \(\|\mu\| < \infty\). Since a norm convergent sequence has a compact absolutely closed convex hull, we get \(\|\mu\| < \infty\) (standard argument). Fix an \(x \in E\) and an \(H \in \mathcal{Y}^\infty(K, X, K)\), \(H\) absolutely convex and closed with pointwise topology. \(H\) is \(\sigma(C_b(X), M_\infty(X))\)-compact [2], [3]. From this it easily follows \(H \otimes x\) is \(\sigma(F, F')\)-compact and absolutely convex and so \(\mu_x \in M_\infty(X)\) which implies that \(\mu \in M_\infty(X, E')\) [2].

**Theorem 2.** If \(X\) is a complete metric space, then \(\beta = \beta_\infty = \beta_0\).

**Proof.** Since \(X\) is a complete metric space, \(M_\infty(X, E) = M_r(X, E') = M_\infty(X, E')\). Also \(\beta\) and \(\beta_\infty\) are both Mackey [2]. From this we get \(\beta = \beta_\infty\). Let \(P\) be an absolutely convex compact subset of \((F', \sigma(F', F))\), where \(F = C_b(X, E)\) and \(F' = M_\infty(X, E')\). This means \(|P|\) is relatively compact in \((M_r(X), \sigma(M_r(X), C_b(X)))\) [2, Proof of Theorem 3.7]. When \(E = K\) and \(X\) is a complete metric space, \(\beta = \beta_0\) [7, Theorem 5.8(a)] and so \(|P|\) is \(\beta_0\)-equicontinuous. Thus, given \(\varepsilon > 0\), there exists a compact \(K \subset X\) such that \(\|\mu\| < \varepsilon\), \(\forall \mu \in P\). This proves \(P\) is \(\beta_0\)-equicontinuous [3, Lemma 2].

**Theorem 3.** Let \(X\) be a metric space and suppose \(E\) contains a \(\sigma\)-weakly-compact dense subset. Then \((F', \sigma(F', F))\) is an angelic space, where \(F = C_b(X, E)\), \(F' = M_\infty(X, E')\).

**Proof.** Since \((F', \tau(F', F))\) is complete, relative countable compactness implies relative compactness in \((F', \sigma(F', F))\) [5, Theorem 11.2, p. 187]. Let \(\tilde{X}\) be the completion of \(X\) and let \(K_0\) be a relative compact subset of \((F', \sigma(F', F))\) and \(\lambda_0 \in K_1 = \text{closure of } K_0\). Every \(\mu \in M_r(X, E')\) gives rise to \(\tilde{\mu} \in M_r(\tilde{X}, E')\), \(\tilde{\mu}(g) = \mu(g|_X)\), \(\forall g \in C_b(\tilde{X}, E)\). Thus \(\tilde{K}_1\) is compact in \((F'_1, \sigma(F'_1, F_1))\), where \(F_1 = C_b(\tilde{X}, E)\) and \(F'_1 = M_r(\tilde{X}, E')\). Since \(\beta_\infty = \beta = \beta_0\) on \(C_b(\tilde{X}, E)\) and \(\beta_\infty\) is strongly Mackey, there exists an increasing sequence \(\{D_n\}\) of compact subsets of \(\tilde{X}\) such that \(\|\mu\|_{\tilde{X}} = 0\), \(\forall \mu \in \tilde{K}_1\). Also since \(F_2 = C_b(\tilde{X}) \otimes E\) is dense in \((F_1, \sigma(F_1, F'_1))\), the topologies \(\sigma(F'_1, F_1)\) and \(\sigma(F'_1, F'_2)\), restricted to \(\tilde{K}_1\), coincide [2].

Let \(X_1 = \text{cl}(\cup D_n)\) in \(\tilde{X}\), and let \(X_2\) be a compact metric space in which \(X_1\) is densely embedded (note \(X_1\) is a separable metric space). For a \(\tilde{\mu} \in \tilde{K}_1\), \(f \in C(X_2)\) and \(x \in E\), define \(\tilde{\mu} \in M_r(X_2, E')\), \(\tilde{\mu}(f \otimes x) = \tilde{\mu}(f|_X \otimes x)\), where \(f|_X\) is any continuous extension of \(f|_{X_1}\) to \(\tilde{X}\) with \(\|f|_X\| = \|f\|\) (sup norm) (this is possible by the Tietze extension theorem; also \(\tilde{\mu}\) is well defined). Since \(C(X_2)\) is separable in norm topology and \(C(X_2) \otimes E\) separates the points of \(\tilde{K}_1 = \{ \tilde{\mu}: \mu \in K_1\}\), there exists a countable subset \(P_1\) of the unit ball of \(C(X_2)\) such that \(P \otimes E\) separates points of \(\tilde{K}_1\) (note \(P \otimes E = \{ p \otimes x: p \in P, x \in E\}\) [5]). This gives a countable set \(P_2\) in the unit ball of \(C_b(\tilde{X})\) such that \(P_2 \otimes E\) separates points of \(\tilde{K}_1\). So we get a countable set \(P\) in the unit ball of \(C_b(X)\) such that \(P \otimes S\) separates the points of \(K_0\), \(S\) being the closed unit ball of \(E\). Giving \(P\) the discrete topology and \(S\) the topology induced by the weak topology on \(E\), we see that \(P \times S\), with product topology, has a \(\sigma\)-compact
dense subset. Also for any \( f \in C_b(X) \) and \( \mu \in M_{\infty}(X, E') \), \( \mu(f \otimes \cdot) \in E' \) and so every \( \mu \in M_{\infty}(X, E') \) is continuous on \( P \times S \). Let \( \{ \mu_\alpha \} \) be a net in \( K_0 \) such that \( \mu_\alpha \to \lambda_0 \) pointwise on \( C_b(X, E) \). By [4, Theorem 0.1] there exists a sequence \( \{ \mu_n \} \subset \{ \mu_\alpha \} \) such that \( \mu_n \to \lambda_0 \) on \( P \times S \). This implies \( \mu_n \to \lambda_0 \) on \( C_b(X, E) \), since \( K \) is compact and \( P \times S \) separates the points of \( K \). This proves the result.

I am grateful to the referee for useful suggestions.

**REFERENCES**


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