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VECTOR-VALUED CONTINUOUS FUNCTIONS WITH STRICT TOPOLOGIES AND ANGELIC TOPOLOGICAL SPACES

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ABSTRACT. It is proved that if X is a metric space, E a Banach space containing a σ-weakly-compact dense subset, then the space $(M_\tau(X, E'), \sigma(M_\tau(X, E'), C_b(X, E)))$ is angelic, $C_b(X, E)$ being all bounded continuous functions from X into E and $M_\tau(X, E')$ the dual of $C_b(X, E)$ with the strict topology β.

A Hausdorff topological space Y is called angelic if (i) every relatively countably compact subset of Y is relatively compact, and (ii) for any point x in the closure of a relatively compact subset A of Y, there exists a sequence, in A, converging to x [**4**, p. 534]. In this paper X will stand for a completely regular Hausdorff space, E a Banach space over K, the field of real or complex numbers, $C(X, E)$ $(C(X))$ all E-valued (K-valued) continuous functions on X, and $C_b(X, E)$ $(C_b(X))$ all bounded E-valued (K-valued) continuous functions on X. We shall use the notations of [**5**] for locally convex spaces. Also the notations and results from [**2**] will be used. The topologies β, β_0, β_1, β_e (also denoted by β_∞) are defined on $C_b(X)$ in [**9**], [**7**], [**8**]–these topologies are defined for $K = R$, the reals, but naturally extend to $K = \mathbf{C}$, the complex field. β, β_0, β_1, β_∞ are defined on $C_b(X, E)$ in [**1**], [**2**]. If $\mathcal{H}^\infty = \mathcal{H}^\infty(X, E) = \{H \subset C_b(X, E)$: H pointwise equicontinuous and uniformly bounded$\}$, the topology β_∞ is the finest locally convex topology on $C_b(X, E)$ agreeing with pointwise topology on each $H \in \mathcal{H}^\infty$. It is well known that $M_\tau(X, E') = (C_b(X, E), \beta)'$, $M_t(X, E') = (C_b(X, E), \beta_0)'$ and $M_\infty(X, E') = (C_b(X, E), \beta_\infty)'$ (see [**2**]). If X is a metric, $M_\infty(X, E') = M_\tau(X, E')$ and β and β_∞ are both Mackey [**2**]; from this it follows that $\beta = \beta_\infty$ in this case.

If X is a metric space it is proved in [**5**, Theorem 5.3] that $(M_\infty(X), \sigma(M_\infty(X), C_b(X)))$ is angelic. We will extend this result to $M_\infty(X, E')$ for some special Banach spaces E.

THEOREM 1. *$(F', \tau(F', F))$ is complete, τ denoting the Mackey topology of the duality $\langle F, F'\rangle$, where $F = C_b(X, E)$ and $F' = M_\infty(X, E')$.*

PROOF. By Grothendieck completeness theorem [**5**, Theorem 6.2, p. 148], it is enough to prove that any linear form μ on F which is continuous on every

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absolutely convex compact subset $(F, \sigma(F, F'))$ with topology induced by β_∞, belongs to F'. Considering μ: $(C_b(X, E), \|\cdot\|) \to K$, we first prove that $\|\mu\| < \infty$. Since a norm convergent sequence has a compact absolutely closed convex hull, we get $\|\mu\| < \infty$ (standard argument). Fix an $x \in E$ and an $H \in \mathcal{K}^\infty(X, K)$, H absolutely convex and closed with pointwise topology. H is $\sigma(C_b(X), M_\infty(X))$-compact [2], [3]. From this it easily follows $H \otimes x$ is $\sigma(F, F')$-compact and absolutely convex and so $\mu_x \in M_\infty(X)$ which implies that $\mu \in M_\infty(X, E')$ [2].

THEOREM 2. *If X is a complete metric space, then $\beta = \beta_\infty = \beta_0$.*

PROOF. Since X is a complete metric space, $M_t(X, E') = M_\tau(X, E') = M_\infty(X, E')$. Also β and β_∞ are both Mackey [2]. From this we get $\beta = \beta_\infty$. Let P be an absolutely convex compact subset of $(F', \sigma(F', F))$, where $F = C_b(X, E)$ and $F' = M_\infty(X, E')$. This means $|P|$ is relatively compact in $(M_\tau(X), \sigma(M_\tau(X), C_b(X)))$ [2, Proof of Theorem 3.7]. When $E = K$ and X is a complete metric space, $\beta = \beta_0$ [7, Theorem 5.8(a)] and so $|P|$ is β_0-equicontinuous. Thus, given $\varepsilon > 0$, there exists a compact $K \subset X$ such that $|\mu|(X \setminus K) < \varepsilon$, $\forall \mu \in P$. This proves P is β_0-equicontinuous [3, Lemma 2].

THEOREM 3. *Let X be a metric space and suppose E contains a σ-weakly-compact dense subset. Then $(F', \sigma(F', F))$ is an angelic space, where $F = C_b(X, E)$, $F' = M_\infty(X, E')$.*

PROOF. Since $(F', \tau(F', F))$ is complete, relative countable compactness implies relative compactness in $(F', \sigma(F', F))$ [5, Theorem 11.2, p. 187]. Let $\tilde{X}$ be the completion of X and let K_0 be a relative compact subset of $(F', \sigma(F', F))$ and $\lambda_0 \in K_1$ = closure of K_0. Every $\mu \in M_\tau(X, E')$ gives rise to $\tilde{\mu} \in M_\tau(\tilde{X}, E')$, $\tilde{\mu}(g) = \mu(g|_X)$, $\forall g \in C_b(\tilde{X}, E)$. Thus $\tilde{K}_1$ is compact in $(F_1', \sigma(F_1', F_1))$, where $F_1 = C_b(\tilde{X}, E)$ and $F_1' = M_\tau(\tilde{X}, E')$. Since $\beta_\infty = \beta = \beta_0$ on $C_b(\tilde{X}, E)$ and β_∞ is strongly Mackey, there exists an increasing sequence $\{D_n\}$ of compact subsets of $\tilde{X}$ such that $|\mu|(\tilde{X} \setminus \bigcup_{n=1}^\infty D_n) = 0$, $\forall \mu \in \tilde{K}_1$. Also since $F_2 = C_b(\tilde{X}) \otimes E$ is dense in $(F_1, \sigma(F_1, F_1'))$, the topologies $\sigma(F_1', F_1)$ and $\sigma(F_1', F_2)$, restricted to $\tilde{K}_1$, coincide [2].

Let $X_1 = \mathrm{cl}(\bigcup D_n)$ in $\tilde{X}$, and let X_2 be a compact metric space in which X_1 is densely embedded (note X_1 is a separable metric space). For a $\tilde{\mu} \in \tilde{K}_1$, $f \in C(X_2)$ and $x \in E$, define $\tilde{\tilde{\mu}} \in M_\tau(X_2, E')$, $\tilde{\tilde{\mu}}(f \otimes x) = \tilde{\mu}(f_1 \otimes x)$, where f_1 is any continuous extension of $f|_{X_1}$ to $\tilde{X}$ with $\|f_1\| = \|f\|$ (sup norm) (this is possible by the Tietze extension theorem; also $\tilde{\tilde{\mu}}$ is well defined). Since $C(X_2)$ is separable in norm topology and $C(X_2) \otimes E$ separates the points of $\tilde{\tilde{K}}_1 = \{\tilde{\tilde{\mu}}: \mu \in K_1\}$, there exists a countable subset P_1 of the unit ball of $C(X_2)$ such that $P \otimes E$ separates points of $\tilde{\tilde{K}}_1$ (note $P \otimes E = \{p \otimes x: p \in P, x \in E\}$ [5]). This gives a countable set P_2 in the unit ball of $C_b(\tilde{X})$ such that $P_2 \otimes E$ separates points of $\tilde{K}_1$. So we get a countable set P in the unit ball of $C_b(X)$ such that $P \otimes S$ separates the points of K_0, S being the closed unit ball of E. Giving P the discrete topology and S the topology induced by the weak topology on E, we see that $P \times S$, with product topology, has a σ-compact dense subset. Also for any $f \in C_b(X)$ and $\mu \in M_\tau(X, E')$, $\mu(f \otimes \cdot) \in E'$ and so every $\mu \in M_\infty(X, E')$ is continuous on $P \times S$. Let $\{\mu_\alpha\}$ be a net in K_0 such that $\mu_\alpha \to \lambda_0$, pointwise on $C_b(X, E)$. By [**4**, Theorem 0.1] there exists a sequence $\{\mu_n\} \subset \{\mu_\alpha\}$ such that $\mu_n \to \lambda_0$ on $P \times S$. This implies $\mu_n \to \lambda_0$ on $C_b(X, E)$, since K is compact and $P \times S$ separates the points of K. This proves the result.

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