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RATIONAL COBORDISM OPERATIONS

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ABSTRACT. Rational cobordism operations dual to the right action are studied. The integral operations, and their compositions are computed in terms of these operations. Similar results for BP-operations are obtained.

Introduction. We study the action of BP cohomology operations on $BP^*(CP^\infty)$. The action of the Landweber-Novikov operations on $MU^*(CP^\infty)$ is quite simple, so our problem reduces to computing the Quillen idempotent.

In [5], the Quillen map is computed on generators M_n of $MU_*(MU)$, for small values of $*$. (These generators are related to the duals of the Landweber-Novikov operations by the canonical anti-isomorphism.)

We formalize this procedure by introducing rational operations ρ_F, which are related to the right action by duality (2.1). These operations behave well with respect to the Quillen map, so the difficulty is reduced to computing ρ_F on $MUQ^*(CP^\infty)$. This allows us to work entirely in cohomology, where the composition laws for ρ_F are simple. In particular the computation of the action of a composition of operations on $BP^*(CP^\infty)$ is no more difficult than computing the action of a single operation (2.4).

I would like to thank M. Mimura for posing the question which led to this paper.

1. Preliminaries on MU and BP. We adopt the notation of [1], [2] and [6]. MU_* is a polynomial algebra on even dimensional generators.

$$MU^{-*} = MU_* = Z[x_1, x_2, \dots], \quad |x_i| = 2i.$$

$MU_*(MU)$ is a polynomial algebra over MU_*.

$$MU_*(MU) = MU_*[b_1, b_2, \dots], \quad |b_i| = 2i.$$

We may also choose generators M_i which are related to b_i by the canonical anti-isomorphism $c(b_i) = M_i$.

Over the rationals, it is possible to choose generators, m_i for MUQ_* so that the right action is given by

$$\eta_R(m_i) = \sum_{a+d=i} m_a (M)_d^{a+1} \tag{1.1}$$

where $M = \Sigma_{i\geq 0} M_i$ and $(M)_d^{a+1}$ denotes the term of degree $2d$ in $(M)^{a+1}$.

Similarly, for a fixed prime p

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$$BP^{-*} = BP_* = Q_p[v_1, v_2, \dots]$$

where $|v_i| = 2(p^i - 1)$, and Q_p denotes the integers localized at p.

$$BP_*(BP) = BP_*[t_1, t_2, \dots], \qquad |t_i| = 2(p^i - 1).$$

We may choose generators for BPQ_* so that

$$BPQ_* = Q[m_1, m_2, \dots], \qquad |m_i| = 2(p^i - 1)$$

and the right action is given by

$$\eta_R(m_i) = \sum_{a+d=i} m_a t_b^{p^a}. \tag{1.2}$$

The Quillen map ε: $MUQ_p \to BP$ is a map of ring spectra, and induces natural transformations

$$\varepsilon^*: MUQ_p^*(X) \to BP^*(X)$$

and

$$\varepsilon_*: MUQ_{p*}(X) \to BP_*(X).$$

Over the rationals we have, for $X = S^0$

$$\varepsilon_*(m_i) = \begin{cases} m_r, & i = p^r - 1, \\ 0, & i \neq p^r - 1, \text{any } r. \end{cases} \tag{1.3}$$

The ring of cohomology operations in $MU^*(MU)$ is the completed tensor product $MU^* \hat{\otimes} S$ where S is the free group generated by elements S_F. ($F = (f_1, f_2, \dots)$ runs over all sequences of nonnegative integers almost all of which are zero.) $\{S_F\}$ is the basis dual to $\{b^F = b_1^{f_1} b_2^{f_2} \dots\}$. $BP^*(BP)$ is the completed tensor product $BP^* \hat{\otimes} R$ where R is the free Q_p module generated by r_F, the dual basis to t^F.

Let E be a spectrum satisfying the flatness condition [**1**]. For $x \in E_*(X)$, $\psi(x) = \sum e_i \otimes x_i$, $e_i \in E_*(E)$, $x_i \in E_*(X)$, $r \in E^*(E)$, and $u \in E^*(X)$ we have the formula of Adams [**1**, p. 73].

$$\langle r(u), x\rangle = \sum \langle r, e_i\langle u, x_i\rangle\rangle. \tag{1.4}$$

For $X = CP^\infty$, $E = MU$ or BP, $E^*(CP^\infty) = E^*[[u]]$, $u \in E^2(CP^\infty)$. So, in this case the action of $E^*(E)$ on $E^*(X)$ is determined by the coaction, ψ, via (1.4). Furthermore, we may choose u in BP and MU so that $\varepsilon^*(u) = u$.

In the case of $E = MU$ we have

$$S_F(u) = \begin{cases} u^{i+1}, & F = \Delta_i, \\ 0, & F \neq \Delta_i, \end{cases}$$

where $\Delta = (0, \dots, 1, \dots)$, 1 in the ith place. The coaction is given by the formula

$$\psi(u_i) = \sum_{a+d=i} (b)_d^a \otimes u_a \tag{1.5}$$

where $\{u_k\}$ is the basis dual to $\{u^k\}$.

2. Rational operations. The formal properties of the right action dualize to convenient composition laws in cohomology. These operations also behave well with respect to the Quillen map. Specifically let E denote MUQ or BPQ.

THEOREM 2.1. *There are operations $\rho_F^E \in E^{|F|}(E)$ ($|F| = \Sigma\, 2if_i$ for $E = MUQ$, $|F| = \Sigma\, 2(p^i - 1)f_i$ for $E = BPQ$) such that*

(i) $F^*(F) = F^* \hat{\otimes} P$ *where P is the Q vector space generated by* $\{\rho_F^E\}$.

(ii) $\Delta(\rho_F^E) = \Sigma_{F_1+F_2=F}\rho_{F_1}^E \otimes \rho_{F_2}^E$.

(iii)

$$\rho_F^E(m^G) = \begin{cases} 1, & F = G, \\ 0, & F \neq G, \end{cases}$$

where $m^G = m_1^{g_1} m_2^{g_2} \dots$.

(iv) $\rho_F^E \circ \rho_D^E = 0$ *if* $F \neq (0, 0, \dots) = (0)$, $\rho_{(0)}^E \circ \rho_D^E = \rho_D^E$.

PROOF. We prove 2.1 for $F = MUQ$. The proof for BPQ is similar.

Set d_k equal to $\eta(m_k)$. (1.1) implies

$$MUQ_*(MU) = MUQ_*[d_1, d_2, \dots],$$

then $d^F = \eta(m^F)$ is an MUQ_* basis for $MUQ_*(MU)$. Define $\{\rho_F = \rho_F^{MU}\}$ to be the basis of $MUQ^*(MU)$ dual to $\{d^F\}$. (i) and (ii) are immediate. From (1.4), with $X = S^0$ we have

$$\langle \rho_F(m^G), 1 \rangle = \langle \rho_F, d^G \rangle$$

and (iii) follows.

To prove (iv) we use the formula

$$\psi(d^F) = 1 \otimes_{MUQ_*} d^F \qquad ([\mathbf{1}, \text{page } 64]).$$

Hence

$$\langle \rho_F \circ \rho_D, d^G \rangle = \langle \rho_F, 1 \cdot \langle \rho_D, d^G \rangle \rangle$$

which is 0 unless $F = (0, 0, \dots)$. For $F = (0, 0, \dots)$

$$\langle \rho_{(0)} \circ \rho_D, d^G \rangle = \begin{cases} 0, & G \neq D, \\ 1, & G = D, \end{cases}$$

and (iv) follows.

REMARKS. (1) $\rho_{(0)}^E$ is not $1 \in E^0(E)$ in fact the counit, ε satisfies $\varepsilon(d^F) = m^F$. This implies $1 = \Sigma m^F \rho_F$.

(2) 2.1 generalizes to ring spectra satisfying the following conditions.

(a) $\pi_*(E) = 0$, $* < 0$.

(b) $\pi_*(E)$ is a finitely generated vector space over the rationals, with $\pi_0(E) = Q$, generated by the unit.

Then, with respect to a basis $\{x_i\}$ (with $1 = x_0$) for E_*, one may choose generators ρ_i, dual to $\eta(x_i)$. This follows from the collapsing of the Atiyah-Hirzebruch spectral sequence. The ρ_i's satisfy (i), (iii) and (iv).

If E_* is a polynomial algebra then we may choose generators satisfying (ii).

We now restrict our attention to BPQ_* for the remainder of this section, and denote ρ_F^{BP} by ρ_F.

Let $F = (f_1, f_2, \ldots)$, $G = (g_1, g_2, \ldots)$ be sequences of nonnegative integers, almost all of which are zero. Define elements $W_{F,G} \in BPQ^*$ by the formula

$$\sum W_{F,G} t^G = \prod \Bigl(\sum_{a+b=n} m_a t_b^{p^a} \Bigr)^{f_n} = d^F.$$

PROPOSITION 2.2. $r_G = \sum_F W_{F,G} \rho_F$.

PROOF. This is dual to (1.2).

For $F = (f_1, f_2, \ldots)$, and a prime p define

$$p(F) = f_1 + pf_2 + p^2 f_3 + \cdots, \qquad s(F) = (f_2, f_3, \ldots).$$

If $G = (g_1, g_2, \ldots)$, then the binomial coefficient $\binom{F}{G}$ is defined to be

$$\binom{f_1}{g_1}\binom{f_2}{g_2} \cdots.$$

For $I \subset BPQ^*$, (I) denotes the ideal $(I) \cdot BPQ^*(BP) \subset BPQ^*(BP)$. With this notation, we may specialize 2.2.

COROLLARY 2.3. (a) $r_{(n)} = \sum_{p(F)=n} \binom{F+G}{F} m^{s(F)+G} \rho_{F+G}$ *where* $(n) = (n, 0, 0, \ldots)$.

(b) $r_F = \rho_F \bmod(m_1, m_2, \ldots)$.

Theorem 2.1 implies we can write compositions of operations, $r_{(n)}$ directly in terms of the basis $\{\rho_F\}$, and thereby obtain relations of the Adem type.

Let $N = (n_k, \ldots, n_2, n_1)$ be a sequence of nonnegative integers. We define

$$r^N = r_{(n_k)} \circ \cdots \circ r_{(n_2)} \circ r_{(n_1)}.$$

For $i = 1, \ldots, k$ let $F_i = (f_1^i, f_2^i, \ldots)$ be a sequence of almost all zero, nonnegative integers. Let T denote $(F_k, \ldots, F_1)$. Fix a prime p. Then we define $p(T) = (p(F_k), \ldots, p(F_1))$, $|T| = F_k + \cdots + F_1$. For $G = (g_1, g_2, \ldots)$

$$C(G, T) = \prod_{q=0}^{k-1} \binom{G + F_1 + S(\Sigma_{i=1}^q F_i) - \Sigma_1^q F_i}{F_{q+1}}$$

where, by convention, $\Sigma_1^0 = 0$. With this notation, we have the following.

PROPOSITION 2.4. $r^N = \sum_{p(T)=N} C(G, T) m^{(G+F_1+S|T|-|T|)} \rho_{G+F_1}$.

PROOF. This is a straightforward induction using (2.1) and (2.3).

EXAMPLES.

(a) $$r_{(1)} \overbrace{\circ \cdots \circ}^{s} r_{(1)} = s! r_{(s)} \quad \bmod(v_1, v_2, \ldots).$$

(b) $$r_{(1,1)} = r_{(2)} \circ r_{(p)} + \binom{p+2}{2}\binom{p}{2} r_{(p+2)} \quad \bmod(v_1, v_2, \ldots).$$

(c) If $m, n < p$ then $r_{(m)} \circ r_{(n)} = r_{(n)} \circ r_{(m)}$. (Compare [6].)

PROOF. For part (a), T must be the sequence $((1), \ldots, (1))$. So $S(|T|) = 0$ and $\Sigma_1^q F_i = (q)$. The only term in $r^{(1, \ldots, 1)}$ not zero $\operatorname{mod}(m_1, \ldots)$ is the term with $G = (s-1)$. In this case, $C(G, T) = s!$. Part (a) now follows from 2.3 (b), and the fact that $(v_1, v_2, \ldots) \otimes Q = (m_1, m_2, \ldots)$ [3]. Parts (b) and (c) are similar.

REMARKS. (1) R. Zahler has obtained an Adem relation for compositions of the elements $r_{(p^i)}$.

(2) The ideal $(v_1, v_2, \ldots)$ is not invariant [4]. So (a) and (b) are only correct on classes $x \in BP^*(X)$, not on classes defined $\operatorname{mod}(v_1, v_2, \ldots)$.

3. Applications to CP^∞. The behavior of ρ_F with respect to the Quillen map is particularly simple. Let Δ_i denote $(0, \ldots, 1, 0, \ldots)$ with 1 in the ith place.

PROPOSITION 3.1. *Suppose $H = \Sigma f_r \Delta_{p^r-1}$ and $\rho_H^{MU}(u) = \Sigma x_{i,H} u^i$, $x_{i,H} \in MUQ^*$. Then $\rho_F^{BP} = \Sigma \varepsilon_*(x_{i,H}) u^i$, where ε_* is as in* (1.3), *and u is defined in* §1.

PROOF.

$$\psi^{MU}(u_k) = \Sigma x_{i,H} d^H \otimes u_1 + (\text{other terms}).$$

By the naturality of the coaction, the naturality of η, and (1.3)

$$\psi^{BP}(u_k) = \Sigma \varepsilon(x_{i,H}) d^F \otimes u_1 + (\text{other terms})$$

where (other terms) does not involve u_1. 3.1 now follows from (1.4).

We now consider operations on $BP^*(CP^n) = BP^*[u]/u^{n+1} = 0$. In particular we consider $r_{(n-k)}$ acting on u^k. In this case we need only compute $r_{(n-k)} \operatorname{mod}(v_1, v_2, \ldots)$ i.e. $\rho_{(n-k)}^{BP}$.

By 3.1 we must compute $\rho_{(n-k)\Delta_{p-1}}^{MU}$. By the Cartan formula it suffices to compute $\rho_{(n-k)\Delta_{p-1}}^{MU}(u)$. For the remainder of this section ρ_* will denote ρ_*^{MU}.

We wish to compute the coefficients $x_{i,l} \in MUQ^*$ in the expansion

$$\rho_{l\Delta_{p-1}}(u) = \Sigma_i x_{i,l} u^i, \qquad \left(|x_{i,l}| = 2\left[l(p-1) - i + 1\right]\right).$$

Set $y_l = \Sigma_i x_{i,l}$, $Y(z) = \Sigma y_l z^l$ (z an indeterminate), and $m = 1 + m_1 + m_2 + \cdots$.

THEOREM 3.2. *y_l is recursively determined by the formula*

$$z(Y(z))^p + Y(z) - m = 0.$$

In particular, for $p = 2$, $Y(z) = (-1 + \sqrt{1 + 4mz})/2z$.

PROOF.

$$y_l = \sum_i \left\langle \rho_{l\Delta_{p-1}}(u), u_i \right\rangle = \sum_i \left\langle \rho_{l\Delta_{p-1}}, (b)_{i-1}^1 \right\rangle.$$

Hence $b = \Sigma y_l d^{l\Delta_{p-1}} + (\text{other terms})$. From (1.1), $\eta(m) = \Sigma m_a (M)^{a+1}$. Applying the canonical anti-isomorphism to both sides, we obtain $m = \Sigma \eta(m_a)(b)^{a+1}$. For $l > 0$,

$$0 = \langle \rho_{l\Delta_{p-1}}, m \rangle = \left\langle \rho_{l\Delta_{p-1}}, b + b^p \eta(m_{p-1}) \right\rangle$$

and for $l = 0$,

$$m = \langle \rho_{l\Delta_{p-1}}, b + b^p\eta(m_{p-1})\rangle.$$

So we have the recursion formula

$$m = \left(\sum y_l d^{l\Delta_{p-1}}\right) + \left(\sum y_l d^{l\Delta_{p-1}}\right) d^{\Delta_{p-1}}.$$

Setting $d^{\Delta_{p-1}}$ equal to the indeterminate z proves 3.2.

We conclude with some examples for $p = 2$.

Let $R = \sum_{l \geq 0} r_{(l)}$. The Cartan formula implies

$$R(u^k) = R(u)^k \quad \text{in } BP^*(CP^n).$$

$$\rho_{(l)}^{MU}(u) = \sum_i x_{i,l} u^i = x_{l+1,l} u^{l+1} \quad \mathrm{mod}(m_1, m_2, \ldots)$$

and

$$Y(z) = \sum y_l z^l = \sum x_{l+1,l} z^l = \frac{-1 + \sqrt{1 + 4z}}{2z} \quad \mathrm{mod}(m_1, m_2, \ldots).$$

From 3.1 we have for the prime 2

$$R(u^k) = u^k \left(\frac{-1 + \sqrt{1 + 4u}}{2u} \right)^k \quad \mathrm{mod}(v_1, v_2, \ldots) \tag{3.3}$$

where $R = r_{(0)} + r_{(1)} + \cdots$.

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