PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 69, Number 1, April 1978

ON THE RELATIONS BETWEEN SOME RATE-OF-GROWTH CONDITIONS

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ABSTRACT. We discuss the implications and nonimplications between four rate-of-growth properties of sets useful in certain areas of recursion theory; all nonimplications are established within the boolean algebra generated by the recursively enumerable sets.

If S is an infinite set of natural numbers, we denote by p_S that strictly increasing function (the so-called *principal function of* S) from the set N of all natural numbers into N whose range is S. In the present paper, S and T (with or without subscripts) always denote *infinite subsets of* N. For any function f, δf denotes the domain of f. Σ_1^0 denotes the class of all recursively enumerable subsets of N, while Π_1^0 denotes $\{S|N - S \in \Sigma_1^0\}$. By a *d.r.e. set*, we mean one which is the difference of two elements of Σ_1^0. $\mathbf{S}$ denotes the Turing degree of S, and $\bar{S}$ denotes the complement of S in N. Let $\langle \varphi_i \rangle$ be some standard recursive enumeration of the partial recursive functions of one argument. As usual, μ denotes the least number operator.

We wish to catalog the various implications and nonimplications among the universal quantifications of the following four "rate-of-growth" conditions which have been studied in [1], [2], [5], [6], and various other places in the recursion-theoretic literature; in the case of each *non*implication, we shall locate a counterexample within one or another familiar subclass of the $\exists\forall \cap \forall\exists$ level of the arithmetical hierarchy.

$[\mathbf{D}_i(\mathbf{S})]$ $\quad (\exists m)(\forall n \geqslant m)[n \in \delta\varphi_i \Rightarrow p_S(n) > \varphi_i(n)];$

$[\mathbf{D}_i^*(\mathbf{S})]$ $\quad (\exists m)(\forall n \geqslant m)[p_S(n) \in \delta\varphi_i \Rightarrow p_S(n+1) > \varphi_i(p_S(n))];$

$[\mathbf{D}_i^{**}(\mathbf{S})]$ $\quad S \subseteq \delta\varphi_i \Rightarrow [D_i^*(S)];$

$[\mathbf{UH}_i(\mathbf{S})]$ $\quad \varphi_i$ total $\Rightarrow [D_i^*(S)]$.

Let $D(S)$ mean $(\forall i)[D_i(S)]$; similarly for the notations "$D^*(S)$", "$D^{**}(S)$", "$UH(S)$". (We have chosen the notation "UH" since the condition $UH(S)$ has several times been referred to in the literature as *uniform hyperimmunity of* S; in the other cases, "D" is for *domination*.)

We begin by stating a result from [1] which is just a bit weaker than one of the facts we shall need:

Received by the editors April 6, 1977.
AMS (MOS) subject classifications (1970). Primary 02F99.
Key words and phrases. Retraceable, cohesive, Σ_1^0.

LEMMA 1 (DEGTEV). *Let M be a maximal Σ_1^0 set. Then $UH(\overline{M})$.*

(Lemma 1 is also mentioned, but not proved, in [3]. In point of fact, Degtev *asserts* in [1] that $D^*(\overline{M})$ holds; his proof, however, stops just short of showing it, the stopping point being $UH(\overline{M})$. In the next lemma, we carry the matter one easy step further.)

Recall that an infinite set $C \subseteq N$ is called *cohesive* if there is no set $W \in \Sigma_1^0$ such that both $C \cap W$ and $C \cap \overline{W}$ are infinite (so that, in particular, *maximal* Σ_1^0 sets are just those which have infinite, cohesive complements).

LEMMA 2. *Let C be a cohesive set such that $C \subseteq \overline{M}$ holds for some maximal element of Σ_1^0. Then $D^*(C)$.*

PROOF. It is very easily seen that $(\forall S)(\forall T)[(D^*(S) \,\&\, T \subseteq S) \Rightarrow D^*(T)]$; hence, it is enough to show that $D^*(\overline{M})$ holds. Now, given φ_i, the cohesiveness of $\overline{M}$ implies that either $\overline{M} \cap \delta\varphi_i$ or $\overline{M} - \delta\varphi_i$ is finite. If $\overline{M} \cap \delta\varphi_i$ is finite, then $[D_i^*(\overline{M})]$ holds "vacuously". If, on the other hand, $\overline{M} - \delta\varphi_i$ is finite, then (by reason of the "Reduction Theorem") there is a *total* recursive function φ_j such that φ_j and φ_i agree on $\overline{M} \cap \delta\varphi_i$. By Lemma 1, $[D_j^*(\overline{M})]$. Hence, since φ_j extends φ_i on $\overline{M}$, $[D_i^*(\overline{M})]$. Thus $D^*(\overline{M})$, and the lemma is proved.

LEMMA 3 [7]. *There exists a maximal Σ_1^0 set M such that* $\mathbf{M} < \mathbf{0}'$.

As we noted at the beginning of the proof of Lemma 2, D^* is a hereditary property; this is true also of D and UH. The situation with respect to D^{**} is quite different, as our proof of Proposition 6, based on the next lemma, will show. (For background material regarding *retraceability*, see [2] or [5].)

LEMMA 4. *Let S be an infinite retraceable set such that $\neg\, D^*(S)$. Then there is a Σ_1^0 set C such that $S \cap C$ is infinite & $\neg\, D^{**}(S \cap C)$.*

PROOF. Let g be a partial recursive function which retraces S, and let $\varphi = \varphi_{i_0}$ be such that $\neg\,[D_{i_0}^*(S)]$. We assume, w.l.o.g., that $g(x) \leqslant x$ for all $x \in \delta g$. Let g^s, φ^s denote, respectively, the sets of pairs belonging to g, φ after s steps in some fixed recursive enumeration of *all* pairs in g, φ (with exactly one pair entering each of g, φ at each step of the enumeration). We shall enumerate C, along with a partial recursive function Ψ having domain C, in stages, as follows.

Stage 0. Set $C^0 = \Psi^0 = \varnothing$; then proceed to Stage 1.

Stage $s + 1$. For each x, let

$$D_x^s = \{z \mid x \in \delta\varphi^s - \delta\Psi^s \,\&\, x < z \leqslant \varphi^s(x) \,\&\, \langle z, x\rangle \in g^s\}.$$

Let

$$E^s = \Big\{z|(\exists t < s)(\exists x)\Big[\langle z, x\rangle \in g^s \\ \& x \in \delta\varphi^t - \delta\Psi^t \,\&\, D_x^t \neq \varnothing \,\&\, x < z \leqslant \varphi^t(x) \\ \& (\forall y < x)\big[y \notin \delta\varphi^t - \delta\Psi^t \vee D_y^t = \varnothing\big] \,\&\, z \notin C^s\Big]\Big\}.$$

If there is no x such that $x \in \delta\varphi^s - \delta\Psi^s \,\&\, D_x^s \neq \varnothing$, set $C^{s+1} = C^s \cup E^s$ and $\Psi^{s+1} = \Psi^s \cup \{\langle w, 0\rangle | w \in E^s - \delta\Psi^s\}$; then proceed to Stage $s + 2$. Otherwise, let $x_0 = (\mu x)[x \in \delta\varphi^s - \delta\Psi^s \,\&\, D_x^s \neq \varnothing]$, and define:

$$C_0^{s+1} = C^s \cup \{x_0\} \cup D_{x_0}^s;$$

$$\Psi_0^{s+1} = \Psi^s \cup \{\langle x_0, \varphi^s(x_0)\rangle\};$$

$$C^{s+1} = C_0^{s+1} \cup E^s;$$

$$\Psi^{s+1} = \Psi_0^{s+1} \cup \{\langle w, 0\rangle | w \in C^{s+1} - \delta\Psi_0^{s+1}\}.$$

Then proceed to Stage $s + 2$.

We define $C = \bigcup_s C^s$, $\Psi = \bigcup_s \Psi^s$. Clearly, C is Σ_1^0 and Ψ is a recursively enumerable set of pairs such that $(\forall s)[\Psi^s \subseteq \Psi^{s+1}]$. Since, by a trivial induction on s, each Ψ^s is seen to be a function, we have that Ψ is a partial recursive function. Obviously $\delta\Psi = C$; moreover, if $x < y \,\&\, \{x, y\} \subseteq C \cap S \,\&\, g(y) = x \,\&\, x = p_S(n)$, then $y = p_S(n + 1)$. The lemma will therefore be proved if we can justify the following claim: *there is a sequence* $\{\langle x_i, y_i\rangle\}$ *of pairs such that* $(\forall i)[x_i < y_i < x_{i+1} \,\&\, \{x_i, y_i\} \subseteq S \cap \delta\Psi \,\&\, g(y_i) = x_i \,\&\, \Psi(x_i) \geqslant y_i]$. Suppose we have found the first n_0 terms, $\langle x_0, y_0\rangle, \ldots, \langle x_{n_0-1}, y_{n_0-1}\rangle$, of such a sequence. (If $n_0 = 0$, we are starting from scratch.) Let $s_0 = (\mu s)$[all x_i and y_i, $i < n_0$, belong to $\delta\Psi^s$]. Let $\langle x, y\rangle$ be the lexicographically least pair such that: $\{x, y\} \subseteq S$, $n_0 > 0 \Rightarrow x > g(x) > y_{n_0-1}$, $\{x, y\} \cap \delta\Psi^{s_0} = \varnothing$, $x < y$, $g(y) = x$, $x \in \delta\varphi - \delta\varphi^{s_0}$, and $\varphi(x) \geqslant y$. Let $w_0 = (\mu w)[x \in \delta\varphi^w]$. We claim that $x \in \delta\Psi$. For let $t_0 = (\mu t)[\{x, y\} \subseteq \delta g^t]$, and suppose that $x \notin \delta\Psi$. Then, at every stage $t \geqslant \max\{w_0, t_0\}$, we have $x \in \delta\varphi^t - \delta\Psi^t \,\&\, D_x^t \neq \varnothing$. But this clearly yields a contradiction, since then x must enter $\delta\Psi$ no later than during stage $\max\{w_0, t_0\} + x_0 + 1$. Thus, $x \in \delta\Psi$. But then, as a trivial induction on s shows, we have either $\Psi(x) = \varphi(x)$ or $\Psi(x) = 0$. If $\Psi(x) = \varphi(x)$, let $z_0 = (\mu z)[x \in \delta\Psi^z]$; then either $y \in \delta\Psi \,\&\, \Psi(y) = \varphi(y)$ or else y enters $\delta\Psi$ via membership in $D_x^s \cup E^s$ for some $s \geqslant z_0$. But then we can define $x_{n_0} = x$, $y_{n_0} = y$. If, on the other hand, $\Psi(x) = 0$, then, as is clear from the construction, we must have $g(x) \in \delta\Psi \,\&\, \Psi(g(x)) = \varphi(g(x)) \geqslant x$; so, in this case, we can define $x_{n_0} = g(x)$, $y_{n_0} = x$. By induction, then, the required sequence $\{\langle x_i, y_i\rangle\}$ exists and the lemma is proved.

We are now ready to present our "catalog".

Proposition 1. $D^*(S) \Rightarrow D^{**}(S) \Rightarrow UH(S)$.

Proof. Obvious, from definitions.

Proposition 2. *If S is regressive, then* $D^*(S) \Rightarrow D(S)$.

PROOF. It is clear that any regressive set satisfying condition D^* is in fact *retraceable*. Now use [5, proof of Theorem 3.2].

Proposition 2 might seem a bit strange at first sight, since $[D_i(S)]$ involves the action of φ_i on $\bar{S}$ while $[D_i^*(S)]$ does not. The next proposition redresses the intuitive balance.

PROPOSITION 3. *There exists a Π_1^0 set S such that $D^*(S) \,\&\, \neg D(S)$.*

PROOF. As shown in [8], $D(S) \Rightarrow \mathbf{S} > \mathbf{0}'$. Applying Lemma 3, let S be a cohesive Π_1^0 set such that $\mathbf{S} < \mathbf{0}'$. Then $\neg D(S)$. On the other hand, $D^*(S)$ holds by Lemma 2.

PROPOSITION 4. *There exists a retraceable Π_1^0 set S such that $D(S) \,\&$ $\neg\, UH(S)$.*

PROOF. By (for instance) [5, Theorems 3.1 and 3.2], there is a retraceable Π_1^0 set T such that $D(T)$ holds. By [6, Theorem 4.1], there is a second retraceable Π_1^0 set R such that $p_R \circ p_T$ is the principal function of a set S for which $\neg\, UH(S)$. But, the condition D is (as is very easily seen) preserved under compositional injection; and, the composition of principal functions of two infinite retraceable Π_1^0 sets is again an infinite retraceable Π_1^0 set. S therefore verifies our proposition. (Easy direct constructions also are available for proving Proposition 4.)

PROPOSITION 5. *There exists a retraceable Π_1^0 set S such that $D^{**}(S)$ & $\neg\, D^*(S)$.*

PROOF. By [2], [4], and [7], let S be a retraceable Π_1^0 set such that $\mathbf{S} < \mathbf{0}'$ & $D^{**}(S)$. By [8] plus Proposition 2, we have $\neg\, D^*(S)$.

PROPOSITION 6. *There exists an infinite d.r.e. set S such that $UH(S)$ & $\neg D^{**}(S)$.*

PROOF. Applying Proposition 5, let S_0 be a retraceable Π_1^0 set such that $\neg D^*(S_0) \,\& D^{**}(S_0)$. Applying Lemma 4, let C be a Σ_1^0 set such that $S_0 \cap C$ is infinite $\& \neg D^{**}(S_0 \cap C)$. Since UH is a hereditary condition, and since UH and D^{**} are equivalent *for Π_1^0 sets* (using the "Reduction Theorem"), we see that $S = S_0 \cap C$ verifies the proposition.

Several fairly obvious questions occur in connection with the foregoing results:

QI. Is there a cohesive set C such that $\neg\, D^*(C)$?

QII. Is there a *complete* maximal Σ_1^0 set M such that $\neg\, D(\bar{M})$?

QIII. If S is Π_1^0, can the set $S \cap C$ of Lemma 4 (and hence the set S of Proposition 6) be required to be retraceable (or even, merely, regressive)? In an earlier version of this paper, we claimed this could be done. The referee, however, spotted a formidable gap in the proof; retraceability was lost during repairs.

QIV. Is Lemma 4 a nonvacuous assertion? That is, is there an example of an infinite set S such that $\neg D^*(S)$ & $(\forall C \in \Sigma_1^0)[S \cap C$ infinite $\Rightarrow D^{**}(S \cap C)]$?

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