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NOTE ON THE NEVANLINNA PROXIMITY FUNCTION

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ABSTRACT. Let λ be a positive function on $(0, \infty)$ with $\lim_{r\to\infty}\lambda(r) = \infty$, and A an arbitrary set of capacity zero. An example is given of a meromorphic function f for which $m(r, a) \to \infty$, $r \to \infty$, whenever $a \in A$, and $T(r, f) = O[(\log r)^2\lambda(r)]$, $r \to \infty$.

1. Introduction. Given f meromorphic in the complex plane, define $m(r, a)$, $N(r)$, $n(r)$ and $T(r)$ in the usual way relative to f (see for example [3]). In this paper we consider the following problem:

If f is a meromorphic function satisfying certain growth restrictions, for how large a set A can

$$\lim_{r\to\infty} m(r, a) = \infty \text{ whenever } a \in A. \tag{1.1}$$

It is well known that A must have capacity zero. We note that the corresponding problem for entire functions of a given order has been completely solved by Drasin and Weitsman [2]. In [1] Damodaran proved

THEOREM. *Let A be an arbitrary set of capacity zero. Given $\lambda(r)$ tending to infinity as $r \to \infty$, there exists a meromorphic function which satisfies* (1.1) *and such that*

$$T(r) = O\big(\lambda(r)(\log r)^3\big), \qquad r \to \infty.$$

Moreover, for any meromorphic function satisfying

$$T(r) = O\big((\log r)^2\big), \qquad r \to \infty,$$

the set A can have at most one element.

The above theorem leaves open the precise requirements on A for orders of growth between $O[(\log r)^2]$ and $O[\lambda(r)(\log r)^3]$. In this note we complete the solution of the above problem by showing

THEOREM 1. *Let λ and A be as in the previous theorem. Then there exists a meromorphic function f which satisfies* (1.1) *and such that*

$$T(r) = O\big[\lambda(r)(\log r)^2\big], \qquad r \to \infty.$$

2. Proof of Theorem 1. We may assume that λ is nondecreasing, absolutely continuous, $\lambda \geqslant 1$, and $r^{-1/2}\lambda(r)$ nonincreasing for $r \geqslant 1$. Otherwise we

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replace λ by λ_1, where λ_1 has these properties, and $\lambda_1 \leqslant \lambda$ for sufficiently large r. Put

$$\sigma(z) = z\int_0^\infty \lambda'\left(\frac{t}{16}\right)(t+z)^{-1}\,dt.$$

Using the fact that $r^{-1/2}\lambda(r)$ is nonincreasing, it follows easily that

$$k_1\lambda(r/16) \leqslant |\sigma(z)| \leqslant k_2\lambda(r/16), \tag{2.1}$$

when $|z| = r$, $r \geqslant 16$, and $|\arg z| \leqslant \pi/2$. Here k_1 and k_2 are positive constants.

Let $(\nu_n)_1^\infty$ be a nondecreasing sequence of positive integers. Given positive integers k and n, where $1 \leqslant k \leqslant \nu_n$, let

$$E'_{k,n} = \left\{z\colon \tfrac{7}{8}2^n \leqslant |z| \leqslant \tfrac{9}{8}2^{n+1}, |\arg z - (4k-1)/16\nu_n| \leqslant 1/12\nu_n\right\},$$

$$E_{k,n} = \left\{z\colon 2^n \leqslant |z| \leqslant 2^{n+1}, |\arg z - (4k-1)/16\nu_n| \leqslant 1/24\nu_n\right\}$$

when n is even, and

$$E'_{k,n} = \left\{z\colon \tfrac{7}{8}2^n \leqslant |z| \leqslant \tfrac{9}{8}2^{n+1}, |\arg z + (4k-1)/16\nu_n| \leqslant 1/12\nu_n\right\},$$

$$E_{k,n} = \left\{z\colon 2^n \leqslant |z| \leqslant 2^{n+1}, |\arg z + (4k-1)/16\nu_n| \leqslant 1/24\nu_n\right\}$$

when n is odd.

The proof of Theorem 1 is based upon the following lemma.

LEMMA 1. *Let $\zeta_{k,n}$, $1 \leqslant k \leqslant \nu_n$, $n = 1, 2, \ldots$, be a sequence of complex numbers with $|\zeta_{k,n}| \leqslant \sqrt{\sigma(2^n)}$, $n = 1, 2, \ldots$. If $\nu_n \leqslant \log \sigma(2^n)$, $n = 1, 2, \ldots$, then there exist a meromorphic function f and a positive integer n_0 such that*

$$|f(z) - \zeta_{k,n}| \leqslant \sigma(2^n)^{-1/4}, \qquad z \in E_{k,n},$$

for $1 \leqslant k \leqslant \nu_n$ and $n \geqslant n_0$. Moreover,

$$T(r) = O\left[\lambda(r)(\log r)^2\right], \qquad r \to \infty.$$

PROOF. Let k, n, be positive integers, $1 \leqslant k \leqslant \nu_n$, and put $d_n = 2^n/(1000\nu_n)$. It is easily seen that $\partial E'_{k,n}$ can be covered by no more than $10002^n/d_n$ disks:

$$\Delta(\zeta_m, r_m) = \{\zeta\colon |\zeta - \zeta_m| \leqslant r_m\}, \qquad \zeta_m \in \partial E'_{k,n},$$

with disjoint interiors and $r_m \leqslant d_n/4$. Let

$$Q(\zeta, z) = -\sum_{p=0}^{N} (\zeta - \zeta_m)^p (z - \zeta_m)^{-(p+1)}$$

for ζ in $\Delta(\zeta_m, r_m)$, where $N = N(n)$ is to be specified later. We note that

$$|Q(\zeta, z)| \leqslant (N+1)(d_n)^N |z - \zeta_m|^{-(N+1)} \tag{2.2}$$

when $|z - \zeta_m| \leqslant d_n$, and ζ is in $\Delta(\zeta_m, r_m)$. Also

$$|Q(\zeta, z) - (\zeta - z)^{-1}| \leqslant 4^{-N}|z - \zeta_m|^{-1}, \tag{2.3}$$

when $|z - \zeta_m| \geqslant d_n$, and ζ in $\Delta(\zeta_m, r_m)$. We remark that for fixed z, $Q(\cdot, z)$ is well defined on $\partial E'_{k,n}$, except for a finite number of $\zeta \in \partial E'_{k,n}$, since the disks $\Delta(\zeta_m, r_m)$ have disjoint interiors. Put

$$g_{k,n}(z) = \frac{1}{2\pi i} \int_{\partial E'_{k,n}} \sigma(\zeta) Q(\zeta, z)\, d\zeta,$$

$$h_{k,n}(z) = \frac{1}{2\pi i} \int_{\partial E'_{k,n}} \sigma(\zeta)(\zeta - z)^{-1}\, d\zeta.$$

Let $\xi_{k,n}$ be a sequence of complex numbers with $|\xi_{k,n}| \leqslant \sqrt{\sigma(2^n)}$ for $1 \leqslant k \leqslant \nu_n$ and $n = 1, 2, \ldots$. In the sequel c denotes a positive constant, not necessarily the same at each occurrence. Given z with $|z| \geqslant 64$, choose p a positive integer so that $2^p \leqslant |z| < 2^{p+1}$. If $n \leqslant p - 3$, then since ζ_m is in $\partial E'_{k,n}$, we have $|\zeta_m| \leqslant 2^{n+2} \leqslant |z|/2$. Consequently,

$$|z - \zeta_m| \geqslant |z|/2 \geqslant 2^{p-1} \geqslant d_n.$$

Using (2.3) it follows that

$$\begin{aligned} |g_{k,n}(z) - h_{k,n}(z)| &\leqslant c2^n\sigma(2^{n+2})4^{-N}\sum_m |z - \zeta_m|^{-1} \\ &\leqslant c2^{n-p}\nu_n 4^{-N}\sigma(2^{n+2}) \end{aligned} \tag{2.4}$$

since there are at most $c\nu_n$ terms in the sum and each is $\leqslant 2^{1-p}$. We now let $N(n) = [5 \log \sigma(2^{n+2})] + 1$, where [ ] denotes the greatest integer function. Clearly, $4^{-N}\sigma(2^{n+2})^5 \leqslant 1$. Since $\nu_n \leqslant \log \sigma(2^n)$, and $|\xi_{k,n}| \leqslant \sqrt{\sigma(2^n)}$, it follows from (2.4) that

$$\begin{aligned} &\sum_{n=1}^{p-3} 2^{-n/2}\left(\sum_{k=1}^{\nu_n} |\xi_{k,n}|\, |g_{k,n}(z) - h_{k,n}(z)| \right) \\ &\qquad \leqslant c \sum_{n=1}^{p-3} 2^{-n/2}(2^{n-p}) \leqslant c2^{-p/2}. \end{aligned} \tag{2.5}$$

Now if dist $(z, \partial E'_{k,n}) \geqslant d_n$ for each $n \geqslant p - 2$ and $1 \leqslant k \leqslant \nu_n$, then from (2.3) we have

$$\begin{aligned} |g_{k,n}(z) - h_{k,n}(z)| &\leqslant c2^n\sigma(2^{n+2})(4^N d_n)^{-1} \\ &\leqslant c\nu_n\sigma(2^{n+2})4^{-N}. \end{aligned} \tag{2.6}$$

Hence,

$$\begin{aligned} &\sum_{n=(p-2)}^{\infty} 2^{-n/2}\left(\sum_{k=1}^{\nu_n} |\xi_{k,n}|\, |g_{k,n}(z) - h_{k,n}(z)| \right) \\ &\qquad \leqslant c \sum_{n=p-2} 2^{-n/2} \leqslant c2^{-p/2}. \end{aligned} \tag{2.7}$$

Let

$$F(z)=\sum_{n=1}^{\infty}2^{-n/2}\left(\sum_{k=1}^{\nu_n}\zeta_{k,n}g_{k,n}(z)\right),\quad G(z)=\sum_{n=1}^{\infty}2^{-n/2}\left(\sum_{k=1}^{\nu_n}g_{k,n}(z)\right),$$
$$f(z)=F(z)/G(z).$$

The series defining F and G converge uniformly on compact subsets of the complex plane, as follows from (2.5), (2.7), and the fact that $h_{k,n}(z)=0$, for z in a compact subset, except for a finite number of pairs (k, n).

If $z \in E_{j,p}$, then

$$|f(z)-\zeta_{j,p}|=|G(z)|^{-1}|F(z)-\zeta_{j,p}G(z)|$$

and dist $(z, \partial E'_{k,n}) \geqslant d_n$ for $1 \leqslant k \leqslant \nu_n$, $n = 1, 2, \ldots$. Since $h_{k,n}(z)=0$ when $(k, n) \neq (j, p)$, and $h_{j,p}(z)=\sigma(z)$, $z \in E_{j,p}$, it follows from (2.5) and (2.7) with $|\xi_{k,n}|=1$, that $|G(z)-2^{-p/2}\sigma(z)| \leqslant c2^{-p/2}$. Hence,

$$|G(z)| \geqslant 2^{-p/2}\big[|\sigma(z)|-c\big],\qquad z \in E_{j,p}. \tag{2.8}$$

Also,

$$|F(z)-\zeta_{j,p}G(z)| \leqslant \sum_{\substack{n=1\\ n\neq p}}^{\infty}\left(\sum_{k=1}^{\nu_n}|\zeta_{k,n}-\zeta_{j,p}|\,|g_{k,n}(z)|\right)2^{-n/2}$$
$$+\,2^{-p/2}\sum_{\substack{k=1\\ k\neq j}}^{\nu_p}|\zeta_{k,p}-\zeta_{j,p}|\,|g_{k,p}(z)|.$$

Using (2.5) and (2.7), we get

$$|F(z)-\zeta_{j,p}G(z)| \leqslant c(|\zeta_{j,p}|+1)2^{-p/2} \leqslant c\sqrt{\sigma(2^p)}\,2^{-p/2}.$$

From (2.8) and (2.1), we now conclude that

$$|f(z)-\zeta_{j,p}| \leqslant \frac{c\left(\sqrt{\sigma(2^p)}\,\right)2^{-p/2}}{2^{-p/2}(|\sigma(z)|-c)} \leqslant \sigma(2^p)^{-1/4}$$

whenever $z \in E_{j,p}$ and $p \geqslant n_0$.

To conclude the proof of Lemma 1, we estimate $T(r, f)$. Since $T(r, f) \leqslant T(r, F) + T(r, 1/G) \leqslant T(r, F) + T(r, G) + O(1)$, it suffices to estimate $T(r, F)$ and $T(r, G)$. Recall for given positive integers k, n, $1 \leqslant k \leqslant \nu_n$, that $\partial E'_{k,n}$ was covered by at most $c2^n/d_n = c\nu_n$ disks $\Delta(\zeta_m, r_m)$. Moreover, $g_{k,n}$ has a pole of order $N(n)+1 \leqslant 6\log\sigma(2^{n+2})$, at ζ_m. Since $\nu_n < \log\sigma(2^n)$, it follows that $g_{k,n}$ has at most $c[\log\sigma(2^{n+2})]^2$ poles all in $\{z\colon 2^{n-1} \leqslant |z| \leqslant 2^{n+2}\}$. If $2^p \leqslant r < 2^{p+1}$, it follows from (2.1) that

$$n(r, F) = n(r, G) \leqslant c\sum_{n=1}^{p+2}\big[\log\sigma(2^{n+2})\big]^2\nu_n \leqslant cp\big[\log\sigma(2^{p+4})\big]^3$$
$$\leqslant c(\log r)(\log\lambda(r))^3 \leqslant c(\log r)\lambda(r).$$

Hence,

$$N(r, F) = N(r, G) \leqslant c(\log r)^2\lambda(r). \tag{2.9}$$

To estimate $m(r, F)$, we note from (2.5) and (2.7) for a given z with $2^p \leqslant |z| \leqslant 2^{p+1}$, that

$$|F(z)| \leqslant c + \sum_{n=p-2}^{p+2} \sum_{k=1}^{\nu_n} |\zeta_{k,n}|\,|g_{k,n}|(z).$$

Since $|\zeta_{k,n}| \leqslant \sqrt{\sigma(2^{n+2})}$, it follows that

$$\log^+|F(z)| \leqslant c + \log \sigma(2^{p+4}) + \log \nu_{p+2} + \sum_{n=p-2}^{p+2} \sum_{k=1}^{\nu_n} \log^+|g_{k,n}|(z).$$

Hence, for $2^p \leqslant r < 2^{p+1}$,

$$m(r, F) \leqslant c + 2\log^+ \sigma(2^{p+4}) + \sum_{n=p-2}^{p+2} \sum_{k=1}^{\nu_n} m(r, g_{k,n}). \tag{2.10}$$

To estimate $m(r, g_{k,n})$, $2^p \leqslant r < 2^{p+1}$, we note from the definition of $g_{k,n}$ that $g_{k,n}(z) = \Sigma_m R_m(z)$, where

$$R_m(z) = \frac{1}{2\pi i} \int_{\Delta(\zeta_m, r_m) \cap \partial E'_{k,n}} \sigma(\zeta) Q(\zeta, z) d\zeta,$$

and the sum has at most $c\nu_n$ terms. From (2.2) we have

$$|R_m(z)| \leqslant cN2^n\sigma(2^{n+2})(d_n)^N|z - \zeta_m|^{-(N+1)},$$

when $|z - \zeta_m| \leqslant d_n$ and from (2.3), $|R_m(z)| \leqslant c\sigma(2^{n+2})d_n$, when $|z - \zeta_m| \geqslant d_n$. Hence,

$$\log^+|R_m(z)| \leqslant c + \log N + n\log 2 + 2\log\sigma(2^{n+2}) + (N+1)\log d_n + (N+1)\log^+(1/|z - \zeta_m|).$$

Since $N \leqslant 6\log\sigma(2^{n+2})$ and $d_n \leqslant 2^n$, it follows that

$$m(r, R_m) \leqslant cn\log\sigma(2^{n+2}).$$

Consequently,

$$m(r, g_{k,n}) = m\left(r, \sum_m R_m\right) \leqslant c\nu_n + \sum_m m(r, R_m)$$
$$\leqslant c\nu_n + c(\nu_n)n\log\sigma(2^{n+2}) \leqslant c\left[\log\sigma(2^{n+2})\right]^2 n.$$

Using (2.10), we conclude

$$m(r, F) \leqslant cp\left[\log\sigma(2^{p+4})\right]^3 \leqslant c(\log r)\lambda(r).$$

Similarly, $m(r,G) \leqslant c(\log r)\lambda(r)$. From (2.9) we now deduce

$$\max\left[T(r, F), T(r, G)\right] \leqslant c(\log r)^2\lambda(r).$$

This completes the proof of Lemma 1. Theorem 1 now follows from Lemma 1 and an argument of Drasin and Weitsman [2, §7]. We sketch their proof when A is a compact set of capacity zero. In this case there is a logarithmic potential P of the form:

$$P(z) = -\sum_{j=1}^{\infty} \alpha_j \log|z - a_j|,$$

which is infinite on A. Here $a_j \in A$, $j = 1, 2, \ldots, (a_i \neq a_j,\ i \neq j)$, and $\sum_{j=1}^{\infty} \alpha_j = 1$. Let f be as in Lemma 1. Then it is possible to choose the sequences (ν_n) and $(\zeta_{k,n})$ in Lemma 1 so that for each positive integer j and $\varepsilon_j > 0$, there corresponds an $r_j = r_j(\varepsilon_j)$ with the following property:

$$|f(re^{i\theta}) - a_j| \leqslant \varepsilon_j, \qquad \theta \in E_j(r), r \geqslant r_j,$$

where $E_j(r)$ has Lebesgue measure $\geqslant c\alpha_j$. Given N a positive integer, choose ε_j, $1 \leqslant j \leqslant N$, so small that the corresponding sets $E_j(r)$ are disjoint for $r \geqslant \max\{r_j\colon 1 \leqslant j \leqslant N\}$. If $r \geqslant \max\{r_j\colon 1 \leqslant j \leqslant N\}$,

$$\begin{aligned} 2\pi m(r, a) &\geqslant \sum_{j=1}^{N} \int_{E_j(r)} \log^+|1/\big(f(re^{i\theta}) - a\big)|\, d\theta \\ &\geqslant c \sum_{j=1}^{N} \alpha_j \log^+\big(1/\left(\varepsilon_j + |a - a_j|\right)\big). \end{aligned}$$

Letting first $r \to \infty$ through a certain sequence, and then $\varepsilon_j \to 0$, $1 \leqslant j \leqslant N$, we get

$$2\pi \liminf_{r\to\infty} m(r, a) \geqslant c \sum_{j=1}^{N} \alpha_j \log^+|1/(a - a_j)|.$$

Letting $N \to \infty$, it follows for $a \in A$ that

$$2\pi \liminf_{r\to\infty} m(r, a) \geqslant cP(a) = +\infty.$$

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