PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 69, Number 1, April 1978

NONHOMOGENEITY OF PRODUCTS OF PREIMAGES AND π-WEIGHT

ERIC K. van DOUWEN

ABSTRACT. We prove a general nonhomogeneity result which implies among others

(1) if X is a homogeneous Hausdorff space, then $|X| \leq 2^{\pi(X)}$;

(2) no power of $\beta(\omega) - \omega$, or of $\beta Q - Q$ or of $\beta R - R$ is homogeneous.

1. Introduction. A space X is *homogeneous* if for every two points x, y of X there is a homeomorphism from X onto itself which maps x onto y. A *π-base* for a space X is a family $\mathfrak{B}$ of nonempty open sets such that each nonempty open set of X includes a member of $\mathfrak{B}$; the *π-weight* of a space X, $\pi(X)$, is $\omega \cdot \min\{|\mathfrak{B}|: \mathfrak{B}$ is a π-base for $X\}$.

In this paper we present a technique, inspired by an idea of Frolík, which is useful for showing that certain spaces are not homogeneous. Roughly speaking, Frolík, showed that a space is not homogeneous by showing that discrete C^*-embedded sequences of points do not cluster in the same way at all points, [$\mathbf{F}_1$], [$\mathbf{F}_3$]. We consider instead completely arbitrary sequences of members of some suitable family of subsets. We make this precise in §2, where we formulate a simple criterion for nonhomogeneity. The contrapositive of this criterion immediately leads to the following quite unexpected result.

1.1. THEOREM. *If X is a homogeneous Hausdorff space, then $|X| \leq 2^{\pi(X)}$.*

Actually we are interested in proving nonhomogeneity. Our main result, Theorem (4.1), implies that if X is Hausdorff and $|X| > 2^{\pi(X)}$, then no power of X, or of certain preimages of X, is homogeneous. For example, no power of $\beta N - N$ is homogeneous (this answers a question of Murray Bell), and no power of $\beta Q - Q$ is homogeneous. A special version of our main result is Theorem (5.1), which deals with high powers. It implies among others that if P is the product of more than 2^{ω} homeomorphs of $\beta Q - Q$, then it is extremely easy to find two points p and q in P such that no homeomorphism of P onto itself maps p onto q: just let p be a point all coordinates of which are equal, and let q be a point having more than 2^{ω} pairwise distinct coordinates.

Received by the editors January 29, 1977 and, in revised form, June 10, 1977.

AMS (MOS) subject classifications (1970). Primary 54A25, 54B10, 54C10, 54F99; Secondary 54D30, 54D40.

Key words and phrases. Nonhomogeneity, π-weight, product, preimage, way to cluster.

The organization of this paper is as follows. In §2 we formulate our criterion for nonhomogeneity, in §3 we collect some lemmas, needed to apply the criterion for proving our main result. The main result is in §4, together with applications. §5 deals with high powers, and in §6 we comment on related techniques and collect questions.

A *cardinal* is an initial ordinal, and *ordinal* is the set of smaller ordinals. We use κ, λ and μ to denote cardinals; we always assume $\kappa \geqslant \omega$. A κ-*sequence* is a function with domain κ. We give all cardinals the discrete topology. $U(\kappa)$ is the space of uniform ultrafilters on κ, or, equivalently,

$$U(\kappa) = \{p \in \beta(\kappa): |V \cap \kappa| = \kappa \text{ for every neighborhood } V \text{ of } p\}.$$

If A and B are sets, ${}^AB$ is the set of functions from A to B. If B also is a space, and $A \neq \varnothing$, then ${}^AB$ gets the usual product topology. So ${}^\kappa 2$ is a *space*, 2^κ is *a cardinal*. Also, if $x \in {}^AB$ and $S \subseteq A$, then the projection of x into ${}^SB$ is $x|S$, the restriction of x to S. The image and inverse image of A under a map f are denoted by $f^{\to}A$ and $f^{\leftarrow}A$, respectively.

The *density*, d(X), and *weight*, w(X), of a space X are defined as usual, [J]; note that by convention d(X) $\geqslant \omega$ and w(X) $\geqslant \omega$ for all spaces X. Q is the space of rationals, R the space of reals.

I am indebted to the referee for carefully reading this paper and for pointing out that my original argument in (4.7) was incomplete; this has forced me to discover (3.3) and (4.1(c)).

2. All possible ways to cluster. Let X be a space, let $\kappa \geqslant \omega$ be a cardinal and let $\mathcal{I}$ be a family of subsets of X. Given $x \in X$ and $\phi \in {}^\kappa\mathcal{I}$, we are interested in "the way ϕ clusters at x", $w(x, \phi)$. We make this precise by defining

$$w(x, \phi) = \{a \subseteq \kappa: x \in \mathrm{Cl} \bigcup \{\phi(\alpha): \alpha \in a\}\}.$$

Then $W(x, \kappa, \mathcal{I}) = \{w(x, \phi): \phi \in {}^\kappa\mathcal{I}\}$ is the set of all possible ways κ-sequences in $\mathcal{I}$ cluster at x.

2.1. Definition. A family $\mathcal{I}$ of subsets of a space X is called an *invariant* family if $h^{\to}I \in \mathcal{I}$ for every $I \in \mathcal{I}$ and every homeomorphism h of X onto itself.

It should be clear that if X is homogeneous, then for every invariant family $\mathcal{I}$, for every κ and for every x and y, $W(x, \kappa, \mathcal{I}) = W(y, \kappa, \mathcal{I})$. A particularly useful special form of the contrapositive of this statement yields the following result.

2.2. Criterion. The space X is *not* homogeneous if there are an invariant family $\mathcal{I}$, a cardinal κ, a $\phi \in {}^\kappa\mathcal{I}$, and a point $p \in X$ such that

$$|W(p, \kappa, \mathcal{I})| < |\{w(x, \phi): x \in X\}|.$$

Proof. Find $x \in X$ such that $w(x, \phi) \notin W(p, \kappa, \mathcal{I})$. Then $W(p, \kappa, \mathcal{I}) \neq W(x, \kappa, I)$. $\square$

3. Tools for proving nonhomogeneity. If one wants to prove that a space X is not homogeneous, with the use of (2.2), one has two tasks to perform for a suitable κ and a suitable invariant family $\mathcal{G}$.

Task 1. Find $\phi \in^{\kappa} \mathcal{G}$ such that $\{w(x, \phi): x \in X\}$ has big cardinality;

Task 2. Find $p \in X$ such that $W(p, \kappa, \mathcal{G})$ has small cardinality.

In the cases we care to consider, 2^{κ} is small and $(2^{\kappa})^{+}$ is big. The following invariant families of subsets of a space are of interest for us:

$\mathcal{K}(X)$: the cozero-sets of X;

$\mathcal{R}(X)$: the regularly open sets of X;

$\mathcal{T}(X)$: the open sets of X.

(Recall that $U \subseteq X$ is regularly open if $U = \text{Int Cl } U$.)

We first show how the π-weight comes in.

3.1. Lemma. *Let X be a Hausdorff space. Denote $\pi(X)$ by κ. Assume that $\mathcal{G} = \mathcal{T}(X)$, or that $\mathcal{G} = \mathcal{K}(X)$ and X is completely regular. Then there is $\phi \in^{\kappa} \mathcal{G}$ such that $w(x, \phi) \neq w(y, \phi)$ for any two distinct $x, y \in X$.*

Proof. In each case considered there is a $\phi: \kappa \to \mathcal{G} - \{\emptyset\}$ such that $\{\phi(\alpha): \alpha \in \kappa\}$ is a π-base. Let $x, y \in X$ be distinct. There is an open U in X with $x \in U, y \notin \text{Cl } U$. Then

$$\{\alpha \in \kappa: \phi(\alpha) \subseteq U\} \in w(x, \phi) - w(y, \phi). \quad \square$$

So if $|X| > 2^{\pi(X)}$ we can perform Task 1, with $\kappa = \pi(X)$. In certain cases this is useful in an indirect way, as our next lemmas show.

3.2. Lemma. *Let X and Y be spaces, let $f: X \to Y$ be a continuous surjection. Let κ be arbitrary, let $\phi \in^{\kappa} \mathcal{T}(Y)$ be given. Define $\psi \in^{\kappa} \mathcal{T}(X)$ by $\psi(a) = f^{\leftarrow}\phi(\alpha)$, $\alpha \in \kappa$. Then*

(a): *if f is an open map, $w(x, \psi) = w(f(x), \phi)$ for all $x \in X$;*

(b): *if f is a retraction, $w(y, \psi) = w(y, \phi)$ for all $y \in Y$.*

Proof. (a): Since $f^{\rightarrow}\psi(\alpha) = \phi(\alpha)$ for all $\alpha \in \kappa$, one readily checks that $w(x, \psi) \subseteq w(f(x), \phi)$ since f is continuous, and $w(f(x), \phi) \subseteq w(x, \psi)$ since f is open.

(b): $w(y, \psi) \subseteq w(y, \phi)$ since f is continuous, and $w(y, \phi) \subseteq w(y, \psi)$ since $y \in Y$ and $\psi(\alpha) \cap Y = \phi(\alpha)$ for $\alpha \in \kappa$. $\square$

The following lemma is particularly useful for compact X, for then the condition on the map is automatically satisfied.

3.3. Lemma. *Let X be a regular space which admits a perfect map f onto a space Y. Let $\kappa = \pi(Y)$. Assume that $\mathcal{G} = \mathcal{T}(X)$, or that $\mathcal{G} = \mathcal{K}(X)$ and Y is completely regular. Then there is an $S \subseteq X$ with $|S| = |Y|$ and there is a $\psi \in^{\kappa} \mathcal{G}$ such that $w(p, \psi) \neq w(q, \psi)$ for any two distinct $p, q \in S$.*

Proof. Call f *semi-open* at $x \in X$ if $\text{Int}_Y f^{\rightarrow}U \neq \emptyset$ for every neighborhood U of x in X. We need the following

Claim. For each $y \in Y$ there is an $x \in f^{\leftarrow}\{y\}$ at which f is semi-open.

Suppose this is false for a certain $y \in Y$. Since X is regular and $f^{\leftarrow}\{y\}$ is compact, we can find a finite family $\mathcal{F}$ of closed sets in X such that $\text{Int}_Y f^{\rightarrow}F = \emptyset$ for all $F \in \mathcal{F}$ but $\bigcup \mathcal{F}$ is a neighborhood of $f^{\leftarrow}\{y\}$ in X. Note that $f^{\rightarrow}F$ is closed in Y for each $F \in \mathcal{F}$. So at the one hand $f^{\leftarrow}(\bigcup \mathcal{F})$ has nonempty interior, being a finite union of closed sets with empty interior, and at the other hand $f^{\to}(\bigcup \mathfrak{F})$ is a neighborhood of y since f is a closed map. This contradiction proves the Claim.

Because of the Claim there is an $S \subseteq X$ such that f is semi-open at each point of S and $f|S\colon S \to Y$ is a bijection. There is a π-base $\{\phi(\alpha)\colon \alpha \in \kappa\}$ for Y such that $\phi(\alpha) \in \mathcal{K}(Y)$ if $\mathcal{G} = \mathcal{K}(X)$. We can define a map $\psi \in {}^{\kappa}\mathcal{G}$ by $\psi(\alpha) = f^{\leftarrow}\phi(\alpha)$, $\alpha \in \kappa$.

Let $p, q \in S$ be distinct. Then $f(p) \neq f(q)$, so there is an open U in Y with $f(p) \in U$ but $f(q) \notin \mathrm{Cl}_Y U$, since Y is Hausdorff, being the perfect image of a Hausdorff space. Let $a = \{\alpha \in \kappa\colon \phi(\alpha) \subseteq U\}$. Then $a \in w(p, \psi)$ since f is semi-open at p, but clearly $a \notin w(q, \psi)$. $\square$

We now proceed to Task 2. The key to all our results is the following triviality:

3.4. FACT. *Let $\mathcal{G}$ be a family of subsets of a space X. Then* $|W(x, \kappa, \mathcal{G})| \leqslant |\mathcal{G}|^{\kappa}$ *for all $x \in X$, all κ.* $\square$

So we are interested in estimates for $|\mathcal{G}|$. The following lemma gives some easy known results. Better results are known, see e.g. **[CH]**, but we have no applications, basically because we are really interested in $|\mathcal{G}|^{\kappa}$, with κ equal to the π-weight. For completeness sake we indicate the easy proofs.

3.5. LEMMA (a). $|\mathfrak{R}(X)| \leqslant 2^{\mathrm{d}(X)}$;

(b). $|\mathcal{K}(X)| \leqslant 2^{\mathrm{d}(X)}$;

(c). $|\mathcal{K}(X)| \leqslant \mathrm{w}(X)^{\omega}$ *if X is Lindelöf.*

PROOF. (a): If $D \subseteq X$ is dense, $\mathfrak{R}(X) = \{\mathrm{int\,Cl}\,A\colon A \subseteq D\}$.

(b): Each member of $\mathcal{K}(X)$ is the union of a countable subfamily of $\mathfrak{R}(X)$, so (b) follows from (a), since $\mathrm{d}(X) \geqslant \omega$ by convention.

(c): Let $\mathfrak{B}$ be a base for X. Each member of $\mathcal{K}(X)$ is an F_σ, hence is Lindelöf, hence is the union of some countable subfamily of $\mathfrak{B}$. $\square$

No such estimates are available for $\mathfrak{T}(X)$, of course, but we do not need this.

3.6. LEMMA. *Let x be any point of any space X. Then* $W(x, \kappa, \mathfrak{T}(X)) = W(x, \kappa, \mathfrak{R}(X))$ *for all κ.*

PROOF. Since $\mathfrak{R}(X) \subseteq \mathfrak{T}(X)$, it suffices to prove that $w(x, \phi) \in W(x, \kappa, \mathfrak{R}(X))$ whenever $\phi \in {}^{\kappa}\mathfrak{T}(X)$. Given $\phi \in {}^{\kappa}\mathfrak{T}(X)$, define $\psi \in {}^{\kappa}\mathfrak{T}(X)$ by $\psi(\alpha) = \mathrm{Int\,Cl}\,\phi(\alpha)$, $\alpha \in \kappa$. Then $\psi \in {}^{\kappa}\mathfrak{R}(X)$ since $\mathrm{Int}\,F \in \mathfrak{R}(X)$ for any closed F in X. But if $U, V \subseteq X$ are open, then $U \cap V = \varnothing$ iff $(\mathrm{Int\,Cl}\,U) \cap (\mathrm{Int\,Cl}\,V) = \varnothing$. It follows that $w(x, \psi) = w(x, \phi)$. $\square$

We would be able to handle small powers with the results obtained so far; see the proof of (4.1). The following lemma is what we need to handle big powers.

3.7. LEMMA. *Let κ and λ satisfy $\lambda \geqslant \kappa \geqslant \omega$. Let X be a space, and assume either that* $\mathrm{d}(X) \leqslant \kappa$ *and $\mathcal{G} = \mathfrak{R}$, or that X is compact and $\mathcal{G} = \mathcal{K}$. Then for every $x \in {}^{\lambda}X$ the following is true*:

$$W(x, \kappa, \mathcal{J}(^{\lambda}X)) = \bigcup \{ W(x|a, \kappa, \mathcal{J}(^{a}X)) : a \subseteq \lambda, |a| = \kappa \}.$$

PROOF. For nonempty $a \subseteq \lambda$ let π_a be the projection from $^{\lambda}X$ onto ^{a}X, and write $\mathcal{J}_a$ for $\mathcal{J}(^{a}X)$. We will need the following easy fact:

(α) For any nonempty $a \subseteq \lambda$, if $J \in \mathcal{J}_a$ then $\pi_a^{\leftarrow} J \in \mathcal{J}_\lambda$.

We first observe that (α) and (3.2(a)) immediately imply that $W(x|a, \kappa, \mathcal{J}_a) \subseteq W(x, \kappa, \mathcal{J}_\lambda)$ for all nonempty $a \subseteq \lambda$.

We next show that for any $x \in {}^{\lambda}X$ and $\phi \in {}^{\kappa}\mathcal{J}_\lambda$ there is an $a \subseteq \lambda$ having cardinality κ, such that $w(x, \phi) \in W(x|a, \kappa, \mathcal{J}_a)$. Because of (3.2(a)) it suffices to prove that

(β) For every $\phi \in {}^{\kappa}\mathcal{J}_\lambda$ there is an $a \subseteq \lambda$ with $|a| = \kappa$ and there is a $\psi \in {}^{\kappa}\mathcal{J}_a$ such that $\phi(\alpha) = \pi_a^{\leftarrow}\psi(\alpha)$ for all $\alpha \in \kappa$, i.e. every $\phi \in {}^{\kappa}\mathcal{J}_\lambda$ depends on κ coordinates.

A moment's reflection shows that this follows from

(γ) for every $A \in \mathcal{J}_\lambda$ there is an $a \subseteq \lambda$ with $1 \leqslant |a| \leqslant \kappa$ and there is a $B \in \mathcal{J}_a$ with $A = \pi_a^{\leftarrow}B$, i.e. every member of $\mathcal{J}_\lambda$ depends on at most κ coordinates.

But this is known:

If $\{X_\alpha: \alpha \in \mu\}$ is any family of spaces with $d(X_\alpha) \leqslant \delta$ for each $\alpha \in \mu$, then each member of $\mathcal{R}(\pi_{\alpha\in\mu}X_\alpha)$ depends on at most δ coordinates, **[RS]**. (The argument uses the fact that no pairwise disjoint family of open sets in $\Pi_{\alpha\in\mu}X_\alpha$ has cardinality $> \delta$, hence every regularly open set of $\Pi_{\alpha\in\mu}X_\alpha$ includes a dense union of at most δ basic open sets. The rest is easy.) And if $\{X_\alpha: \alpha \in \mu\}$ is any family of compact spaces, then every continuous real-valued function defined on $\Pi_{\alpha\in\mu}X_\alpha$ depends on at most countably many coordinates; as pointed out in **[EP]**, this is an easy corollary to the Stone-Weierstrass Theorem. (We are interested only in the cozero-sets in $\Pi_{\alpha\in\mu}X_\alpha$. The following elementary proof is available: each cozero-set in $\Pi_{\alpha\in\mu}X_\alpha$ is σ-compact, hence is the union of at most countably many sets of the form $\Pi_{\alpha\in\mu}B_\alpha$, where B_α is an open F_σ in X_α for all $\alpha \in \mu$, and $B_\alpha = X_\alpha$ for all but finitely many α. The rest is easy.) □

3.8. REMARK. Instead of cozero-sets we could have considered zero-sets in this section.

3.9. REMARK. An alternative way of performing Task 1 would be to find $\phi \in {}^{\kappa}\mathcal{J}$ such that $(\mathrm{Cl} \bigcup \{\phi(\alpha): \alpha \in a\}) \cap (\mathrm{Cl} \bigcup \{\phi(\alpha): \alpha \in \kappa - a\}) = \emptyset$ for all $a \subseteq \kappa$, provided X is compact. Indeed, then one can find for each free ultrafilter $\mathcal{F}$ on κ a point $x \in X$ with $w(x, \phi) = \mathcal{F}$, simply by choosing $x \in \bigcap \{\mathrm{Cl} \bigcup \{\phi(\alpha): \alpha \in a\}: a \in \mathcal{F}\}$, and it is well known that there are $2^{2^{\kappa}}$ ultrafilters on κ, **[CN**, 7.4] or **[GJ**, 9.2]. One does not really need compactness, it is sufficient to know that one can pick $p(\alpha) \in \phi(\alpha)$ for $\alpha \in \kappa$ such that $\mathrm{Cl}\{p(\alpha): \alpha \in \kappa\}$ has cardinality greater than 2^{κ}, e.g. because $\mathrm{Cl}\{p(\alpha): \alpha \in \kappa\}$ is compact. This can be used e.g. with $X = \beta Q - Q$.

4. Nonhomogeneity. The following theorem implies (1.1), see also (5.1).

4.1. THEOREM. *If the space X admits a continuous map f onto a Hausdorff* *space Y with $|Y| > 2^{\pi(Y)}$, then no power of X is homogeneous in each of the following cases*:

(a) *f is open or is a retraction and* $\mathrm{d}(X) \leqslant \pi(Y)$;

(b) *f is perfect, X is regular and* $\mathrm{d}(X) \leqslant \pi(Y)$; *or*

(c) *X is compact Hausdorff and* $\mathrm{w}(X) \leqslant 2^{\pi(Y)}$.

PROOF. Denote $\pi(Y)$ by κ, and let $\lambda \geqslant 1$ be arbitrary.

Case (a). Since ${}^{\lambda}X$ admits an open map onto X, it follows from (3.1) and two applications of (3.2) that there is a $\phi \in {}^{\kappa}\mathfrak{T}({}^{\lambda}X)$ such that

$$\left|\left\{w(x, \phi): x \in {}^{\lambda}X\right\}\right| = |Y| > 2^{\kappa}.$$

It remains to find a point $p \in {}^{\lambda}X$ such that $|W(p, \kappa, \mathfrak{T}({}^{\lambda}X))| \leqslant 2^{\kappa}$, or, equivalently, by (3.6), such that $|W(p, \kappa, \mathfrak{R}({}^{\lambda}X))| \leqslant 2^{\kappa}$, for then ${}^{\lambda}X$ is not homogeneous by (2.2).

In $\lambda \leqslant \kappa$ (or even $\lambda \leqslant 2^{\kappa}$), then any $p \in {}^{\lambda}X$ will do. For then $\mathrm{d}({}^{\lambda}X) \leqslant \kappa$ (by the Hewitt-Marczewski-Pondiczery Theorem), so the result follows from (3.4) and (3.5(a)).

If $\lambda \geqslant \kappa$, consider any $p \in {}^{\lambda}X$ which is constant (as a function from λ to X). Then $W(p|a, \kappa, \mathfrak{R}({}^{a}X)) = W(p|\kappa, \kappa, \mathfrak{R}({}^{\kappa}X))$ for all $a \subseteq \lambda$ with $|a| = \kappa$, hence

$$W\left(p, \kappa, \mathfrak{R}({}^{\lambda}X)\right) = W\left(p|\kappa, \kappa, \mathfrak{R}({}^{\kappa}X)\right)$$

by (3.7). But the latter set has cardinality $\leqslant 2^{\kappa}$, by the argument we just gave for the case $\lambda \leqslant \kappa$.

Cases (b) *and* (c). The argument is almost identical; one also should use (3.3) and (3.5(c)). □

4.2. THEOREM. *No power of $\beta Q - Q$ is homogeneous. More generally, if bQ is a compactification of Q with $|bQ| > 2^{\omega}$, then no power of $bQ - Q$ is homogeneous.*

PROOF. First note that $|\beta Q| = 2^{2^{\omega}}$, **[GJ**, 9.3], so the second statement is more general. Since no point of Q has a compact neighborhood, $bQ - Q$ is dense in bQ. Now $\pi(Y) = \pi(X)$ whenever X is regular and Y is dense in X, **[J**, 2.3]. It follows that $\pi(bQ - Q) = \pi(bQ) = \pi(Q) = \omega$. Now apply (4.1). □

That $\beta Q - Q$ is not homogeneous follows from Frolík's result that $\beta X - X$ is not homogeneous if X is not pseudocompact, **[F$_3$]**; see **[vD$_1$]** and **[vD$_2$]** for totally different proofs. The results on powers of $bQ - Q$ and on $\beta Q - Q$ are new. We leave it to the reader to generalize (4.2).

4.3. THEOREM. *Let $\kappa \geqslant \omega$. Then no power of $\beta(\kappa)$, $\beta(\kappa) - \kappa$ or $U(\kappa)$ is homogeneous.*

PROOF. Fix $\kappa \geqslant \omega$. We first note that $\beta(\kappa) - \kappa$ and $U(\kappa)$ can be mapped continuously onto $\beta(\kappa)$. Indeed, there is a continous $f: \beta(\kappa) \to \beta(\kappa)$ such that $|f^{\leftarrow}\{\alpha\}| = \kappa$ for each $\alpha \in \kappa$. Then f maps $U(\kappa) \subseteq \beta(\kappa) - \kappa$ onto $\beta(\kappa)$.

Each of $\beta(\kappa)$, $\beta(\kappa) - \kappa$ and $U(\kappa)$ has weight $\leqslant \mathrm{w}(\beta(\kappa)) = 2^{\kappa} = 2^{\pi(\beta(\kappa))}$.

So the theorem follows from (4.1(c)). □

That $U(\omega) = \beta(\omega) - \omega$ is not homogeneous is due to Frolík, [$\mathbf{F_1}$]. That $U(\kappa)$ and $\beta(\kappa) - \kappa$ are not homogeneous for $\kappa > \omega$ is an easy consequence of Remarks 4 and 1 in Frolík's [$\mathbf{F_2}$].

4.4. THEOREM. *No power of βR or of $\beta R - R$ is homogeneous.*

PROOF. $|\beta R| = 2^{2^\omega}$ [**GJ**, 9.3], but $\pi(\beta R) = \omega$ since R is a dense open subspace of βR. Hence no power of βR is homogeneous by (4.1), since $w(\beta R) = 2^\omega$. We claim that $\beta R - R$ can be mapped continuously onto βR, it then follows from (4.1(c)) that no power of $\beta R - R$ is homogeneous.

Let $\alpha R = R \cup \{\infty\}$ be the one-point compactification of R. Then the subspace

$$S = \{\langle x, y\rangle \in \alpha R \times \beta R: x \in R, y = x \cdot \sin x\}$$

of $\alpha R \times \beta R$ is homeomorphic to R. But clearly $\bar{S} = S \cup \{\infty\} \times \beta R$. It follows that R has a compactification bR with $bR - R$ homeomorphic to βR. There is a continuous map $f: \beta R \to bR$ with $f(x) = x$ for $x \in R$; then $f^{\to}(\beta R - R) = (bR - R)$ [**GJ**, 6.12]. □

That $\beta R - R$ is not homogeneous follows from Frolík's result that $\beta X - X$ is not homogeneous if X is not pseudocompact, [$\mathbf{F_3}$]; see [$\mathbf{vD_3}$] for a totally different proof.

4.5. REMARK. The condition that Y be Hausdorff in (4.1) is essential: Let μ be any infinite cardinal. Define a space M as follows: the underlying set of M is $\mu \times \omega$, and $U \subseteq M$ is open iff $U \supseteq \mu \times (\omega - n)$ for some $n \in \omega$. Then M is a homogeneous T_1-space, $|M| = \mu$ but $\pi(M) = \omega$.

One cannot replace π-weight by density in (4.1): ${}^{(2^\kappa)}2$ is homogeneous but has density $\leqslant \kappa$.

I do not know if the conditions on f are essential in (4.1(a)) and (4.1(b)). (Easy examples show that the conditions on f are essential in (3.2) and (3.3).)

4.6. REMARK. One should compare (1.1) with the fact that $|X| \leqslant 2^{\pi(X)^{c(X)}}$ for every Hausdorff space X, where $c(X)$ is the *cellularity* of X. (Argument: $|\mathfrak{R}(X)| \leqslant 2^{\pi(X)^{c(X)}}$ for every space X, and $|X| \leqslant 2^{|\mathfrak{R}(X)|}$ if X is Hausdorff.)

4.7. REMARK. One can use (4.1(c)) to give an unusual proof of the fact that βX is not dyadic if X is not pseudocompact, [**EP**, Theorem 3]. (Recall that a Hausdorff space Y is *dyadic* if it is the continuous image of ${}^{\mu}2$ for some μ, and that then μ can be taken to be $w(Y)$, [**S**] or [**EP**, Theorem 1].) It suffices to prove that βR is not dyadic, by [**EP**, §2]. Indeed, ${}^{(2^\omega)}2$ is homogeneous, but $w({}^{(2^\omega)}2) = w(\beta R) = 2^\omega = 2^{\pi(\beta R)} = 2^\omega$, and $|\beta R| = 2^{2^\omega}$ [**GJ**, 9.3].

4.8. REMARK. The analogue of (4.2) for compactifications of ω is not true. Indeed, ω has a compactification $b(\omega)$ such that $b(\omega) - \omega$ is homeomorphic to the homogeneous space ${}^{(2^\omega)}2$, by [**E**], since ${}^{(2^\omega)}2$ is separable.

5. High powers. In (4.1) we proved that ${}^{\lambda}X$ is not homogeneous for suitable X by finding suitable κ, $\mathcal{G}$ for which there are p, q with $W(p, \kappa, \mathcal{G}) \neq W(q, \kappa, \mathcal{G})$. This required a cardinality argument. The following theorem is remarkable since it shows that it is extremely easy to find such p and q if λ is sufficiently big; moreover our intuition recognizes p and q as being different. Also, p and q are more different than in (4.1): we will prove that $W(p, \kappa, \mathcal{G}) \neq W(q, \kappa, \mathcal{G})$ by showing that they have different cardinality.

5.1. THEOREM. *Let X be a space which admits a continuous map f onto a Hausdorff space Y with $|Y| > 2^{\pi(Y)}$. Assume that one of the following holds:*

(a) *f is open or is a retraction, and* $d(X) \leqslant \pi(Y)$;

(b) *X is regular, f is perfect and* $d(X) \leqslant \pi(Y)$;

(c) *X is compact Hausdorff and* $w(X) \leqslant 2^{\pi(Y)}$.

Then there is a subset S of X with $|S| = |Y|$ such that the following holds: if $\lambda > 2^{\pi(Y)}$, and p and q are any two points of ${}^{\lambda}X$ such that $p: \lambda \to X$ is constant and $q: \lambda \to X$ takes on at least $(2^{\pi(Y)})^{+}$ values which lie in S, then no homeomorphism of ${}^{\lambda}X$ onto itself maps p onto q.

As in (4.1) we only consider case (a). Denote $\pi(Y)$ by κ. There is by (3.1) an $S \subseteq X$ with $|S| = |Y|$, and there is a $\phi \in {}^{\kappa}\mathfrak{T}(X)$ such that $w(x, \phi) \neq w(y, \phi)$ for any two distinct $x, y \in S$.

Let $\lambda > 2^{\pi(Y)}$, and let p and q be as stated. The proof of (4.1) shows that $W(p, \kappa, \mathfrak{T}({}^{\lambda}X)) \leqslant 2^{\kappa}$. But $w(q(\alpha), \phi) \in W(q, \kappa, \mathfrak{T}({}^{\lambda}X))$ for every $\alpha \in \lambda$ by (3.2). Consequently $|W(q, \kappa, \mathfrak{T}({}^{\lambda}X))| \geqslant |S \cap q^{\to}\lambda| > 2^{\kappa}$. Hence

$$W\big(p, \kappa, \mathfrak{T}({}^{\lambda}X)\big) \neq W\big(q, \kappa, \mathfrak{T}({}^{\lambda}X)\big). \qquad \square$$

5.2. REMARK. In case (a) it is sufficient to assume that $f \circ q: \lambda \to Y$ takes on at least $(2^{\pi(Y)})^{+}$ values. However, in neither case it is sufficient that $q: \lambda \to X$ takes on at least $(2^{\pi(Y)})^{+}$ values: consider the case that $X = \beta\omega \times {}^{(2^{\omega})}2$, $Y = \beta\omega$.

5.3. REMARK. (4.1) follows from (5.1) since a product of homogeneous spaces is homogeneous.

6. Discussion and questions. Frolík considered the way discrete C^*-embedded ω-sequences of one-point sets cluster at the points of a space, [$\mathbf{F}_1$], [$\mathbf{F}_3$]. The idea of using κ-sequences of other than one-point sets, not necessarily for $\kappa = \omega$, is due independently to Comfort and Negrepontis [CN, 16.18], and the present author.

We did not impose any condition on the ϕ's, used in the definition of $W(x, \kappa, \mathcal{G})$ in §2, simply because there is no use for such conditions. Examples of additional conditions are

(1) $\{\phi(\alpha): \alpha \in \kappa\}$ is pairwise disjoint, $\bigcup\{\phi(\alpha): \alpha \in \kappa\}$ is C^*-embedded, as Comfort and Negrepontis do for compact X, with $\mathcal{G} \subseteq \mathfrak{T}(X)$, or the weaker (since $\mathcal{G} \subseteq \mathfrak{T}(X)$) condition considered in (3.8);

(2) $(\mathrm{Cl} \bigcup \{\phi(\alpha): \alpha \in a\}) \cap (\mathrm{Cl} \bigcup \{\phi(\alpha): \alpha \in \kappa - a\}) = \varnothing$ for all $a \subseteq \kappa$. (That (2) suffices in [CN, 16.18] was also observed by Murray Bell, who used this to prove that ${}^{\lambda}U(\omega)$ is not homogeneous for $1 \leqslant \lambda \leqslant \omega$. His question of whether ${}^{\lambda}U(\omega)$ is not homogeneous for all cardinals $\lambda \geqslant 1$ was the original motivation for this paper.) It is important to realize however, that the only

thing (2) does, is to perform what we called Task 1 (provided some ϕ satisfies (2) and X is compact, of course). Indeed, the fact that we did not impose any conditions on the ϕ's was crucial for the discovery of (1.1).

There are several examples of nonhomogeneous spaces which have a homogeneous power. The most famous example is perhaps the closed unit interval I: ${}^{\omega}I$ is the Hilbert cube which is known to be homogeneous, [K]. Another example is $\alpha(\omega)$, the one-point compactification of ω; indeed, ${}^{\omega}\alpha(\omega)$ is homeomorphic to ${}^{\omega}2$. It is quite easy to give an example of a compact space X no power of which is homogeneous: let X be the subspace $[0, 1] \cup \{2\}$ of R; then for any cardinal $\lambda \geqslant 1$ not all components of ${}^{\lambda}X$ have the same cardinality. I am not aware of any example in the literature of a (compact) zero-dimensional space no power of which is homogeneous. This paper supplies several such spaces. A very natural example is $\alpha(\kappa)$, the one-point compactification of κ, for any $\kappa > \omega$. One can show that for all $\lambda \geqslant 1$, some but not all points $x \in {}^{\lambda}\alpha(\kappa)$ have the following property:

There is an uncountable pairwise disjoint family $\mathfrak{U}$ of clopen sets in ${}^{\lambda}\alpha(\kappa)$ such that every neighborhood of x intersects all but finitely many members of $\mathfrak{U}$. (In other words, *disjoint* ω_1-sequences of clopen sets in ${}^{\lambda}\alpha(\kappa)$ do not cluster the same way at all points.)

The reason for the conditions $\mathrm{d}(X) \leqslant \mathrm{d}(Y)$, or X is compact and $\mathrm{w}(X) \leqslant 2^{\pi(Y)}$ in (4.1) was that we had to perform what is called Task 2, in §3. It is not known if they are essential. This leads to the following questions. A positive answer would be very strong indeed.

6.1. *Question*. Is a compact Hausdorff space nonhomogeneous if it can be mapped continuously onto $\beta(\omega)$, or onto $\beta(\kappa)$, or onto a Hausdorff space Y with $|Y| > 2^{\pi(Y)}$?

6.2. *Question*. Is a (compact Hausdorff) space nonhomogeneous if some open subspace can be mapped by an open mapping or a retraction onto $\beta(\omega)$, or onto a Hausdorff space Y with $|Y| > 2^{\pi(Y)}$?

The only thing I know is the following easy proposition (and some modifications), which one can easily prove by considering as invariant family the clopen sets.

6.3. PROPOSITION. *If a compact Hausdorff space X admits a continuous map f onto $\beta(\kappa)$ (or $U(\kappa)$), with the property that $f^{\leftarrow}\{y\}$ is connected for each $y \in \beta(\kappa)$ (or $y \in U(\kappa)$), then X is not homogeneous. (In fact, no power would be homogeneous.)*

Theorem 1.1 suggests the possibility there are more relations between cardinal functions on homogeneous (compact) (Hausdorff) spaces, which are not true without the assumption of homogeneity. I am not aware of any other such relation. It would certainly be worthwhile to further investigate this.

REFERENCES

[CH] W. W. Comfort and A. W. Hager, *Estimates for the number of real-valued continuous* *functions*, Trans. Amer. Math. Soc. **150** (1970), 619–631.

[CN] W. W. Comfort and S. Negrepontis, *The theory of ultrafilters*, Springer, Heidelberg, 1974.

[vD$_1$] E. K. van Douwen, *Why certain Čech-Stone remainders are not homogeneous*, Colloq. Math. (to appear).

[vD$_2$] ______, *Remote points*. (in prep.).

[vD$_3$] ______, *Subcontinua and nonhomogeneity of* $\beta R^+ - R^+$ (in prep.).

[E] R. Engelking, *On the double circumference of Alexandroff*, Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys. **16** (1968), 629–634.

[EP] R. Engelking and A. Pełczyński, *Remarks on dyadic spaces*, Colloq. Math. **11** (1963), 55–63.

[F$_1$] Z. Frolík, *Sum of ultrafilters*, Bull. Amer. Math. Soc. **73** (1967), 87–91.

[F$_2$] ______, *Fixed points of maps of* βN, Bull. Amer. Math. Soc. **74** (1968), 187–191.

[F$_3$] ______, *Nonhomogeneity of* $\beta P - P$, Comment. Math. Univ. Carolinae **8** (1967), 705–709.

[GJ] L. Gillman and M. Jerison, *Rings of continuous functions*, Van Nostrand, Princeton, N. J., 1960.

[J] I. Juhász, *Cardinal functions in topology*, Mathematical Centre Tract **34**, Amsterdam, 1971.

[K] O. H. Keller, *Die Homoimorphie der kompakten konvexen Mengen in Hilbertschen Raum*, Math. Ann. **105** (1931), 748–758.

[RS] K. A. Ross and A. H. Stone, *Products of separable spaces*, Amer. Math. Monthly **71** (1964), 398–403.

[S] N. A. Šanin, *On the product of topological spaces*, Trudy Mat. Inst. Steklov. **24** (1948), 112 pp. (Russian) (cited in [EP]).

INSTITUTE FOR MEDICINE AND MATHEMATICS, MATHEMATICS BUILDING, OHIO UNIVERISTY, ATHENS, OHIO 45701