DECOMPOSITION OF APPROXIMATE DERIVATIVES

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Abstract. It is shown that if \( f: [0, 1] \to \mathbb{R} \) has a finite approximate derivative \( f_{ap} \) everywhere in \([0, 1]\), then there is a sequence of perfect sets \( H_n \), whose union is \([0, 1]\), and a sequence of differentiable functions, \( h_n \), such that \( h_n = f \) over \( H_n \) and \( h'_n = f_{ap} \) over \( H_n \). This result follows from a new, more general theorem relating approximate differentiability and differentiability. Applications of both theorems are given.

In this paper we illustrate a new sense in which an approximate derivative is a derivative. This approach can be used to both clarify known properties of these derivatives and also establish additional properties. Basically, we prove that an approximate derivative can be decomposed, in a way which will be made precise, into a sequence of derivatives. This result will be obtained from the following new theorem relating the concepts of differentiability and approximate differentiability.

Theorem 1. Let \( Q \) be a measurable set and \( E \) a closed subset of the points of density of \( Q \). Suppose \( f: Q \to \mathbb{R} \) is a measurable function possessing a finite approximate derivative at each point of \( E \). Then \( E \) can be expressed as the countable union of closed sets \( E_n \) such that for each \( n \) and each \( x \) in \( E_n \),

\[
E_n = \lim_{y \to x} \frac{f(y) - f(x)}{y - x} = f'_{ap}(x).
\]

Here the notation \( E_n \to x \) means that we approach \( x \) only through the set \( E_n \setminus \{x\} \). At an isolated point of \( E_n \) the conclusion is considered to hold vacuously. It will be shown later that the sets \( E_n \) can be chosen to be perfect sets.

Before the proof we state a lemma. It is a modification of lemmas which can be found in [4] and [5].

Lemma. Let \( Q \) and \( E \) be as stated in Theorem 1. Suppose \( f \) is approximately continuous at every point of \( E \). Let \( n \) be a fixed integer. Define

\[
A_n(x) = \{ y \in Q: |f(y) - f(x)| < n|y - x| \}
\]

and
\( E_n = \{ x \in E : m(A_n \cap I) \geq \frac{3}{4} m(I) \text{ for all intervals I containing } x \text{ with length of } I < 1/n \}. \)

(Here \( m \) denotes Lebesgue measure.) Then

(a) \( E_n \) is a closed set, and

(b) if \( x \) and \( y \) belong to \( E_n \) and \( |x - y| < 1/n \), then \( |f(x) - f(y)| < n|x - y| \).

**Proof.** The proof of (b) is obvious. The proof of (a) is rather lengthy and requires proving first that (b) holds at every two limit points of \( E_n \). Since the proof can be arrived at by modifying appropriately the proofs in [4] or [5] it will not be given here. We proceed instead with the proof of Theorem 1.

**Proof of Theorem 1.** For each \( n \) we apply the above Lemma to get a sequence of closed sets \( E_n \). These \( E_n \) form the desired decomposition of \( E \). To see this let \( x \) belong to \( E \). There is an \( N_1(x) \) such that for \( n > N_1 \), \( A_n(x) \) has density 1 at \( x \). Then there is an \( N_2 > N_1 \) such that for \( n > N_2 \), \( x \) belongs to \( E_n \). Thus \( \bigcup_{1 \leq n \leq \infty} E_n = E \). It remains only to show that \( f \) is differentiable relative to \( E_n \) and differentiates to \( f'_{ap} \). This is more involved.

Let \( n \) be fixed and \( x \) belong to \( E_n \). Let \( x_k \) be a sequence of points of \( E_n \) converging to \( x \). It will not hurt the generality of the argument to assume that \( x = 0 \) and \( f(0) = 0 \). Since \( f \) is approximately differentiable at 0 there is a measurable set \( V \subset Q \) having density 1 at 0 for which

\[ V = \lim_{x \to 0} \frac{f(x)}{x} = f'_{ap}(0). \]

By use of the Lusin-Menchoff Theorem [2], we may assume that \( V \) is a perfect set. There is an \( n^{-1} > \delta_1 > 0 \) such that for all \( 0 < x < \delta_1 \), \( m(V \cap [0, x]) > x2^{-1} \). We assume that \( x_k < \delta_1 \) for all \( k \) and define

\[ L_k = \{ z : 0 < z < x_k \text{ and } m(V \cap [z, x_k]) > 2^{-1}(x_k - z) \}. \]

The set \( L_k \) is not empty since 0 belongs to \( L_k \). Let \( l_k \) be the least upper bound of \( L_k \). If \( l_k = x_k \), select a point \( z_k \) belonging to \( L_k \) with \( x_k > z_k > (1 - 1/k)x_k \). If \( l_k < x_k \) let \( z_k = l_k \). Then in either case

\[ m(V \cap [z_k, x_k]) > \frac{1}{2}(x_k - z_k). \]

For each \( k \) \( x_k \) belongs to \( E_n \), and since the length of \( [z_k, x_k] \) is less than \( n^{-1} \) we have

\[ m(A_n(x_k) \cap [z_k, x_k]) > \frac{3}{4}(x_k - z_k). \]

Hence

\[ m(V \cap A_n(x_k) \cap [z_k, x_k]) > 0. \]

We select another point \( y_k \) from this intersection, strictly between \( z_k \) and \( x_k \). For the sequence \( y_k \) we have

(i) \( \lim_{k \to +\infty} f(y_k)/y_k = f'_{ap}(0) \), and

(ii) \( |f(y_k) - f(x_k)| \leq n(x_k - y_k) \).
Now
\[ \frac{f(x_k)}{x_k} = \frac{f(x_k) - f(y_k)}{x_k - y_k} \left[ 1 - \frac{y_k}{x_k} \right] + \frac{f(y_k)}{y_k} \frac{y_k}{x_k}. \]
Thus from (i) and (ii) above if we show that \( \lim_{k \to +\infty} y_k/x_k = 1 \), then
\[ \lim_{k \to \infty} f(x_k)/x_k = f'_\text{ap}(0). \]
To get this final part we make a few preliminary observations. First we note that \( z_k < y_k < x_k \), so it suffices to prove that \( z_k/x_k \) approaches 1. Next, the sequence \( z_k \) was chosen in two different ways.
For those \( k \) such that \( l_k = x_k \), we have \( z_k > (1 - 1/k)x_k \). Therefore, we need only consider any subsequence of \( k \)'s for which \( l_k < x_k \). In this case \( z_k = l_k \), and we will be finished if we show that \( \lim_{k \to +\infty} l_k/x_k = 1 \).
Let \( 0 < \varepsilon < \frac{1}{4} \) be fixed. There is a \( K \) such that for \( k > K \), \( m(V \cap [0, x_k]) > (1 - \varepsilon)x_k \). For such a \( k \) consider the interval \([x_k - 3\varepsilon x_k, x_k]\). In this interval
\[ m(V \cap [x_k - 3\varepsilon x_k, x_k]) > x_k - \varepsilon x_k - [m(V \cap [0, x_k - 3\varepsilon x_k])] \]
\[ > x_k - \varepsilon x_k - [x_k - 3\varepsilon x_k] = 2\varepsilon x_k = \frac{2}{3} m([x_k - 3\varepsilon x_k, x_k]). \]
Hence,
\[ l_k \geq x_k - 3\varepsilon x_k, \]
which completes the proof.
We note that Theorem 1 could be applied to get a version of the known result by Whitney [6]. However, we proceed to consider functions which are approximately differentiable everywhere in \([0, 1]\).

**Theorem 2.** If \( f: [0, 1] \to \mathbb{R} \) has a finite approximate derivative, \( f'_\text{ap} \), at every point of \([0, 1]\), then there is a sequence of perfect sets \( H_n \) and a sequence of differentiable functions \( h_n \) such that
(i) \( h_n(x) = f(x) \) over \( H_n \),
(ii) \( h'_n(x) = f'_\text{ap}(x) \) over \( H_n \), and
(iii) \( \bigcup_{1 \leq n < \infty} H_n = [0, 1] \).
The sequence \( (h_n, H_n) \) is called a decomposition of \( f \). The corresponding sequence \( (h'_n, H_n) \) is the before-mentioned decomposition of the approximate derivative \( f'_\text{ap} \).

**Proof.** We first obtain \( H_n \). Let \( E_n \) be the sets defined in Theorem 1. We have shown that \( f \) differentiates to \( f'_\text{ap} \), relative to \( E_n \). However, \( E_n \) may have isolated points. We can express \( E_n \) as the union of a perfect set \( P_n \) and a countable set \( C_n \). Let \( C \) be the union, over \( n \), of these \( C_n \). Arrange the elements of \( C \) into a sequence which we also label \( x_n \). For each fixed \( x_n \) we can find a perfect set \( V_n \), having density 1 at \( x_n \), such that
\[ V_n - \lim_{y \to x_n} \frac{f(y) - f(x)}{y - x} = f'_\text{ap}(x_n). \]
We choose a disjoint sequence of intervals \( I_k = [a_k, b_k] \) having the properties that
(1) the point \( x_n \) is not in any \( I_k \),
(2) \( \lim_{k \to +\infty} a_k = x = \lim_{k \to +\infty} b_k \), and
(3) \( m(I_k \cap V_n) > 0 \) for each \( k \).

For each \( k \) it is possible to find an integer \( N(k) \) such that \( m(\mathbb{E}^{N(k)} \cap I_k \cap V_n) > 0 \). For such an \( N(k) \) we select a perfect set \( W_k \) from the intersection. Then we set \( Q_n \) equal to the union, over \( k \), of \( W_k \) and \( \{ x_n \} \). Then \( Q_n \) is perfect. Finally, we set \( H_n = P_n \cup Q_n \). It is clear that the \( H_n \) are perfect sets and, for each \( x \) in \( H_n \),

\[
H_n - \lim_{y \to x} \frac{f(y) - f(x)}{y - x} = f'_\text{ap}(x).
\]

We note that the above argument could be applied in Theorem 1 to make \( E_n \) perfect. At this point we are able to apply the theorem of Petruska and Laczkovich [3]. This theorem guarantees that for each \( N \) it is possible to obtain a differentiable function \( h_n \) such that \( h_n = f \) over \( H_n \). This completes the proof.

It is obvious that the existence of the sets \( H_n \) presents a situation where the Baire category theorem can be usefully employed. We do so. As applications we present two corollaries. The first gives transparent proofs of two known theorems [1].

**Corollary 1.** Let \( f: [0, 1] \to \mathbb{R} \) have a finite approximate derivative \( f'_\text{ap} \) everywhere in \([0, 1]\). Then
(a) there is a dense open set \( U \) such that \( f \) is differentiable on each component of \( U \), and
(b) the function \( f'_\text{ap} \) is Baire 1.

**Proof.** Let \((h_n, H_n)\) and \((h'_n, H'_n)\) be decompositions of \( f \) and \( f'_\text{ap} \).
(a) Let \( H^o_n \) be the interior of \( H_n \). Let \( U \) be the union of the \( H^o_n \). By the Baire category theorem, \( U \) is a dense open subset of \([0, 1]\). Clearly, since \( f = h_n \) over \( H^o_n \), \( f \) is differentiable over every component of \( U \).
(b) Let \( P \) be a perfect set. Let \( P_n = H_n \cap P \). Again an application of the Baire category theorem yields that there is an \( N \) and \((a, b)\) such that \( \emptyset \neq (a, b) \cap P \subseteq P_N \). Then \( f'_\text{ap} = h'_N \) over \( P_N \). Since \( h'_N \) is Baire 1 it has a point of relative continuity in \( P_N \), and, hence, \( f'_\text{ap} \) is Baire 1 on \([0, 1]\).

Basic to the concept of approximate differentiability is the idea that at a point \( x_0 \) we may disregard the behavior of a function over certain "small" sets. Therefore it becomes natural to expect that knowledge of the behavior of \( f \) on a small set, such as nowhere dense sets of measure zero, would not permit the prediction of the values of \( f'_\text{ap} \) over this set. However, the next corollary shows that this is not quite true.

**Corollary 2.** Let \( f: [0, 1] \to \mathbb{R} \) and \( g: [0, 1] \to \mathbb{R} \) be two measurable functions. Suppose \( P \) is any perfect set such that \( f(x) = g(x) \) over \( P \). Suppose, in addition, that at every point of \( P \) \( f \) is approximately differentiable and \( g \) is differentiable. Then there is an open interval \((a, b)\) with \((a, b) \cap P \neq \emptyset \) such that \( f'_\text{ap} = g' \) at every point of \((a, b) \cap P \).
Proof. Let \( h(x) = f(x) - g(x) \) for all \( x \) in \([0, 1]\). Then over \( P \) \( h = 0 \) and \( h_{ap}' = f_{ap}' - g' \). By Theorem 1 there is a sequence of closed sets \( P_n \) such that \( h \) differentiates to \( h_{ap}' \) over \( P \) and \( \bigcup_{1 \leq n < \infty} P_n = P \). Once again the Baire category theorem yields an \((a, b)\) and \( N \) such that \( \emptyset \neq (a, b) \cap P \subset P_N \). Now \( P \) is perfect, and for every \( x \) in \((a, b) \cap P \) we have

\[
P - \lim_{y \to x} \frac{h(y) - h(x)}{y - x} = h_{ap}'(x) = 0 = f_{ap}'(x) - g'(x).
\]

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References


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