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DECOMPOSITION OF APPROXIMATE DERIVATIVES

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ABSTRACT. It is shown that if f: $[0, 1] \to R$ has a finite approximate derivative f'_{ap} everywhere in $[0, 1]$, then there is a sequence of perfect sets H_n, whose union is $[0, 1]$, and a sequence of differentiable functions, h_n, such that $h_n = f$ over H_n and $h'_n = f'_{\text{ap}}$ over H_n. This result follows from a new, more general theorem relating approximate differentiability and differentiability. Applications of both theorems are given.

In this paper we illustrate a new sense in which an approximate derivative is a derivative. This approach can be used to both clarify known properties of these derivatives and also establish additional properties. Basically, we prove that an approximate derivative can be decomposed, in a way which will be made precise, into a sequence of derivatives. This result will be obtained from the following new theorem relating the concepts of differentiability and approximate differentiability.

THEOREM 1. *Let Q be a measurable set and E a closed subset of the points of density of Q. Suppose f: $Q \to R$ is a measurable function possessing a finite approximate derivative at each point of E. Then E can be expressed as the countable union of closed sets E_n such that for each n and each x in E_n,*

$$E_n - \lim_{y \to x} \frac{f(y) - f(x)}{y - x} = f'_{\text{ap}}(x).$$

Here the notation $E_n - \lim_{y \to x}$ means that we approach x only through the set $E_n \setminus \{x\}$. At an isolated point of E_n the conclusion is considered to hold vacuously. It will be shown later that the sets E_n can be chosen to be perfect sets.

Before the proof we state a lemma. It is a modification of lemmas which can be found in [4] and [5].

LEMMA. *Let Q and E be as stated in Theorem 1. Suppose f is approximately continuous at every point of E. Let n be a fixed integer. Define*

$$A_n(x) = \{y \in Q: |f(y) - f(x)| \leqslant n|y - x|\}$$

and

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$$E_n = \{x \in E\colon m(A_n \cap I) \geq \tfrac{3}{4} m(I) \text{ for all intervals } I \text{ containing } x \text{ with length of } I < 1/n\}.$$

(Here m denotes Lebesgue measure.) Then

(a) *E_n is a closed set, and*

(b) *if x and y belong to E_n and $|x - y| < 1/n$, then $|f(x) - f(y)| \leq n|x - y|$.*

PROOF. The proof of (b) is obvious. The proof of (a) is rather lengthy and requires proving first that (b) holds at every two limit points of E_n. Since the proof can be arrived at by modifying appropriately the proofs in **[4]** or **[5]** it will not be given here. We proceed instead with the proof of Theorem 1.

PROOF OF THEOREM 1. For each n we apply the above Lemma to get a sequence of closed sets E_n. These E_n form the desired decomposition of E. To see this let x belong to E. There is an $N_1(x)$ such that for $n > N_1$, $A_n(x)$ has density 1 at x. Then there is an $N_2 > N_1$ such that for $n > N_2$, x belongs to E_n. Thus $\bigcup_{1 \leq n \leq \infty} E_n = E$. It remains only to show that f is differentiable relative to E_n and differentiates to f'_{ap}. This is more involved.

Let n be fixed and x belong to E_n. Let x_k be a sequence of points of E_n converging to x. It will not hurt the generality of the argument to assume that $x = 0$ and $f(0) = 0$. Since f is approximately differentiable at 0 there is a measurable set $V \subset Q$ having density 1 at 0 for which

$$V - \lim_{x \to 0} \frac{f(x)}{x} = f'_{\mathrm{ap}}(0).$$

By use of the Lusin-Menchoff Theorem **[2]**, we may assume that V is a perfect set. There is an $n^{-1} > \delta_1 > 0$ such that for all $0 < x < \delta_1$, $m(V \cap [0, x]) > x2^{-1}$. We assume that $x_k < \delta_1$ for all k and define

$$L_k = \{z\colon 0 \leq z < x_k \text{ and } m(V \cap [z, x_k]) \geq 2^{-1}(x_k - z)\}.$$

The set L_k is not empty since 0 belongs to L_k. Let l_k be the least upper bound of L_k. If $l_k = x_k$, select a point z_k belonging to L_k with $x_k > z_k > (1 - 1/k)x_k$. If $l_k < x_k$ let $z_k = l_k$. Then in either case

$$m(V \cap [z_k, x_k]) \geq \tfrac{1}{2}(x_k - z_k).$$

For each k x_k belongs to E_n, and since the length of $[z_k, x_k]$ is less than n^{-1} we have

$$m(A_n(x_k) \cap [z_k, x_k]) \geq \tfrac{3}{4}(x_k - z_k).$$

Hence

$$m(V \cap A_n(x_k) \cap [z_k, x_k]) > 0.$$

We select another point y_k from this intersection, strictly between z_k and x_k. For the sequence y_k we have

(i) $\lim_{k \to +\infty} f(y_k)/y_k = f'_{\mathrm{ap}}(0)$, and

(ii) $|f(y_k) - f(x_k)| \leq n(x_k - y_k)$.

Now

$$\frac{f(x_k)}{x_k} = \frac{f(x_k) - f(y_k)}{x_k - y_k}\left[1 - \frac{y_k}{x_k}\right] + \frac{f(y_k)}{y_k} \cdot \frac{y_k}{x_k}.$$

Thus from (i) and (ii) above if we show that $\lim_{k\to+\infty} y_k/x_k = 1$, then $\lim_{k\to\infty} f(x_k)/x_k = f'_{\text{ap}}(0)$. To get this final part we make a few preliminary observations. First we note that $z_k < y_k < x_k$, so it suffices to prove that z_k/x_k approaches 1. Next, the sequence z_k was chosen in two different ways. For those k such that $l_k = x_k$, we have $z_k > (1 - 1/k)x_k$. Therefore, we need only consider any subsequence of k's for which $l_k < x_k$. In this case $z_k = l_k$, and we will be finished if we show that $\lim_{k\to+\infty} l_k/x_k = 1$.

Let $0 < \varepsilon < \frac{1}{4}$ be fixed. There is a K such that for $k > K$, $m(V \cap [0, x_k]) > (1 - \varepsilon)x_k$. For such a k consider the interval $[x_k - 3\varepsilon x_k, x_k]$. In this interval

$$\begin{aligned} m(V \cap [x_k - 3\varepsilon x_k, x_k]) &\geqslant x_k - \varepsilon x_k - [m(V \cap [0, x_k - 3\varepsilon x_k])] \\ &\geqslant x_k - \varepsilon x_k - [x_k - 3\varepsilon x_k] = 2\varepsilon x_k = \frac{2}{3} m([x_k - 3\varepsilon x_k, x_k]). \end{aligned}$$

Hence,

$$l_k \geqslant x_k - 3\varepsilon x_k,$$

which completes the proof.

We note that Theorem 1 could be applied to get a version of the known result by Whitney [6]. However, we proceed to consider functions which are approximately differentiable everywhere in [0, 1].

THEOREM 2. *If f: $[0, 1] \to R$ has a finite approximate derivative, f'_{ap}, at every point of* $[0, 1]$, *then there is a sequence of perfect sets H_n and a sequence of differentiable functions h_n such that*

(i) $h_n(x) = f(x)$ *over* H_n,

(ii) $h'_n(x) = f'_{\text{ap}}(x)$ *over* H_n, *and*

(iii) $\bigcup_{1\leqslant n<\infty} H_n = [0, 1]$.

The sequence (h_n, H_n) is called a decomposition of f. The corresponding sequence (h'_n, H_n) is the before-mentioned decomposition of the approximate derivative f'_{ap}.

PROOF. We first obtain H_n. Let E_n be the sets defined in Theorem 1. We have shown that f differentiates to f'_{ap}, relative to E_n. However, E_n may have isolated points. We can express E_n as the union of a perfect set P_n and a countable set C_n. Let C be the union, over n, of these C_n. Arrange the elements of C into a sequence which we also label x_n. For each fixed x_n we can find a perfect set V_n, having density 1 at x_n, such that

$$V_n - \lim_{y\to x_n} \frac{f(y) - f(x)}{y - x} = f'_{\text{ap}}(x_n).$$

We choose a disjoint sequence of intervals $I_k = [a_k, b_k]$ having the properties that

(1) the point x_n is not in any I_k,

(2) $\lim_{k\to+\infty} a_k = x = \lim_{k\to+\infty} b_k$, and
(3) $m(I_k \cap V_n) > 0$ for each k.

For each k it is possible to find an integer $N(k)$ such that $m(E_{N(k)} \cap I_k \cap V_n) > 0$. For such an $N(k)$ we select a perfect set W_k from the intersection. Then we set Q_n equal to the union, over k, of W_k and $\{x_n\}$. Then Q_n is perfect. Finally, we set $H_n = P_n \cup Q_n$. It is clear that the H_n are perfect sets and, for each x in H_n,

$$H_n - \lim_{y\to x} \frac{f(y) - f(x)}{y - x} = f'_{\mathrm{ap}}(x).$$

We note that the above argument could be applied in Theorem 1 to make E_n perfect. At this point we are able to apply the theorem of Petruska and Laczkovich [3]. This theorem guarantees that for each N it is possible to obtain a differentiable function h_n such that $h_n = f$ over H_n. This completes the proof.

It is obvious that the existence of the sets H_n presents a situation where the Baire category theorem can be usefully employed. We do so. As applications we present two corollaries. The first gives transparent proofs of two known theorems [**1**].

Corollary 1. *Let f: $[0, 1] \to R$ have a finite approximate derivative f'_{ap} everywhere in* $[0, 1]$. *Then*

(a) *there is a dense open set U such that f is differentiable on each component of U, and*

(b) *the function f'_{ap} is Baire* 1.

Proof. Let (h_n, H_n) and (h'_n, H_n) be decompositions of f and f'_{ap}.

(a) Let H_n° be the interior of H_n. Let U be the union of the H_n°. By the Baire category theorem, U is a dense open subset of $[0, 1]$. Clearly, since $f = h_n$ over H_n°, f is differentiable over every component of U.

(b) Let P be a perfect set. Let $P_n = H_n \cap P$. Again an application of the Baire category theorem yields that there is an N and (a, b) such that $\varnothing \neq (a, b) \cap P \subset P_N$. Then $f'_{\mathrm{ap}} = h'_N$ over P_N. Since h'_N is Baire 1 it has a point of relative continuity in P_N, and, hence, f'_{ap} is Baire 1 on $[0, 1]$.

Basic to the concept of approximate differentiability is the idea that at a point x_0 we may disregard the behavior of a function over certain "small" sets. Therefore it becomes natural to expect that knowledge of the behavior of f on a small set, such as nowhere dense sets of measure zero, would not permit the prediction of the values of f'_{ap} over this set. However, the next corollary shows that this is not quite true.

Corollary 2. *Let f: $[0, 1] \to R$ and g: $[0, 1] \to R$ be two measurable functions. Suppose P is any perfect set such that $f(x) = g(x)$ over P. Suppose, in addition, that at every point of P f is approximately differentiable and g is differentiable. Then there is an open interval (a, b) with $(a, b) \cap P \neq \varnothing$ such that $f'_{\mathrm{ap}} = g'$ at every point of $(a, b) \cap P$.*

PROOF. Let $h(x) = f(x) - g(x)$ for all x in $[0, 1]$. Then over P $h = 0$ and $h'_{\mathrm{ap}} = f'_{\mathrm{ap}} - g'$. By Theorem 1 there is a sequence of closed sets P_n such that h differentiates to h'_{ap} over P and $\bigcup_{1 \leq n < \infty} P_n = P$. Once again the Baire category theorem yields an (a, b) and N such that $\varnothing \neq (a, b) \cap P \subset P_N$. Now P is perfect, and for every x in $(a, b) \cap P$ we have

$$P - \lim_{y \to x} \frac{h(y) - h(x)}{y - x} = h'_{\mathrm{ap}}(x) = 0 = f'_{\mathrm{ap}}(x) - g'(x).$$

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