A SHORT PROOF OF A GREENE THEOREM

CHUNG-LIE WANG

Abstract. A short and simple proof of an inequality of the Gronwall type is given for a class of integral systems based upon the generalized Gronwall lemma of Sansone-Conti.

Recently David E. Greene [1] used a technically involved iteration in proving the following

Theorem [Greene]. Let $K_1$, $K_2$ and $\mu$ be nonnegative constants and let $f$, $g$ and $h_i$ be continuous nonnegative functions for all $t > 0$ with $h_i$ bounded such that

$$f(t) < K_1 + \int_0^t h_1(s)f(s)\,ds + \int_0^t e^{\mu h_2(s)}g(s)\,ds,$$

$$g(t) < K_2 + \int_0^t e^{-\mu h_3(s)}f(s)\,ds + \int_0^t h_4(s)g(s)\,ds$$

for all $t > 0$. Then there exist constants $c_i$ and $M_i$ such that

$$f(t) < M_1 e^{c_1 t}, \quad g(t) < M_2 e^{c_2 t}$$

for all $t > 0$.

In this note, a short and simple proof of this theorem based upon the generalized Gronwall lemma of Sansone-Conti which is cited from [2] in a restricted form as follows

(Generalized) Gronwall Lemma. For all $t > 0$, let three functions $\lambda$, $\phi$, $u$ be given such that $\lambda$ is summable and nonnegative, $\phi$ is absolutely continuous, and $u$ is continuous. If $u(t) < \phi(t) + \int_0^t \lambda(s)u(s)\,ds$, then

$$u(t) < \int_0^t \phi'(s) \exp\left(\int_s^t \lambda(r)\,dr\right) ds + \phi(0) \exp\left(\int_0^t \lambda(r)\,dr\right).$$

Proof of the theorem. Let $P$ be an upper bound for $h_i$ (the assumption $\mu > 0$ is not necessarily required here), then

$$f(t) < K_1 + P \int_0^t f(s)\,ds + P \int_0^t e^{\mu g(s)}\,ds,$$

$$(1)$$

$$g(t) < K_2 + P \int_0^t e^{-\mu f(s)}\,ds + P \int_0^t g(s)\,ds.$$
Multiplying (1) by $e^{-\mu t}$ and then adding to (2),

$$e^{-\mu t}f(t) + g(t)$$

$$\leq Ke^{-\mu t} + K_2 + \int_0^t P[e^{\mu(s-t)} + 1][e^{-\mu s}f(s) + g(s)]\,ds$$

$$\leq Ke^{-\mu t} + K_2 + \int_0^t 2P[e^{-\mu s}f(s) + g(s)]\,ds.$$  \hspace{1cm} (3)

Applying the lemma to (3)

$$e^{-\mu t}f(t) + g(t)$$

$$\leq \int_0^t (-K_1\mu e^{-\mu t})\exp\left(\int_s^t 2Pdr\right)\,ds + (K_1 + K_2)\exp\left(\int_0^t 2Pds\right)$$

$$= \frac{K_1\mu}{2P + \mu} e^{-\mu t} + \frac{2P(K_1 + K_2) + K_2\mu}{2P + \mu} e^{2Pt}.$$  

The conclusion of the theorem is now clear.

REFERENCES


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF REGINA, REGINA, SASKATCHEWAN, CANADA S4S 0A2