A SHORT PROOF OF A GREENE THEOREM

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Abstract. A short and simple proof of an inequality of the Gronwall type is given for a class of integral systems based upon the generalized Gronwall lemma of Sansone-Conti.

Recently David E. Greene [1] used a technically involved iteration in proving the following

Theorem [Greene]. Let $K_1, K_2$ and $\mu$ be nonnegative constants and let $f, g$ and $h_i$ be continuous nonnegative functions for all $t > 0$ with $h_i$ bounded such that

$$f(t) < K_1 + \int_0^t h_1(s)f(s)\,ds + \int_0^t e^{\mu h_2(s)}g(s)\,ds,$$

$$g(t) < K_2 + \int_0^t e^{-\mu h_3(s)}f(s)\,ds + \int_0^t h_4(s)g(s)\,ds$$

for all $t > 0$. Then there exist constants $c$ and $M_i$ such that

$$f(t) < M_1 e^{ct}, \quad g(t) < M_2 e^{ct}$$

for all $t > 0$.

In this note, is presented a short and simple proof of this theorem based upon the generalized Gronwall lemma of Sansone-Conti which is cited from [2] in a restricted form as follows

(Generalized) Gronwall Lemma. For all $t > 0$, let three functions $\lambda, \phi, u$ be given such that $\lambda$ is summable and nonnegative, $\phi$ is absolutely continuous, and $u$ is continuous. If $u(t) < \phi(t) + \int_0^t \lambda(s)u(s)\,ds$, then

$$u(t) < \int_0^t \phi'(s)\exp\left(\int_s^t \lambda(r)\,dr\right)\,ds + \phi(0)\exp\left(\int_0^t \lambda(r)\,dr\right).$$

Proof of the Theorem. Let $P$ be an upper bound for $h_i$ (the assumption $\mu > 0$ is not necessarily required here), then

$$f(t) < K_1 + P \int_0^t f(s)\,ds + P \int_0^t e^{\mu s}g(s)\,ds,$$  \hspace{1cm} (1)

$$g(t) < K_2 + P \int_0^t e^{-\mu s}f(s)\,ds + P \int_0^t g(s)\,ds.$$  \hspace{1cm} (2)

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Multiplying (1) by \( e^{-\mu t} \) and then adding to (2),

\[
e^{-\mu t}f(t) + g(t)
\]

\[
\leq K_1 e^{-\mu t} + K_2 + \int_0^t P[ e^{\mu(s-t)} + 1 ] [ e^{-\mu t}f(s) + g(s) ] \, ds
\]

\[
\leq K_1 e^{-\mu t} + K_2 + \int_0^t 2P[ e^{-\mu t}f(s) + g(s) ] \, ds.
\]

Applying the lemma to (3)

\[
e^{-\mu t}f(t) + g(t)
\]

\[
\leq \int_0^t (-K_1 \mu e^{-\mu s}) \exp \left( \int_s^t 2Pdr \right) ds + (K_1 + K_2) \exp \left( \int_0^t 2Pds \right)
\]

\[
= \frac{K_1 \mu}{2P + \mu} e^{-\mu t} + \frac{2P(K_1 + K_2)}{2P + \mu} e^{2Pt}.
\]

The conclusion of the theorem is now clear.

**References**


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