THE ISOPERIMETRIC THEOREM
FOR CURVES ON MINIMAL SURFACES

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ABSTRACT. A short proof is given for a sharpened form of the isoperimetric
inequality for curves on minimal surfaces.

By following a line of development used by Sachs [7] in treating inequali-
ties for plane curves, one can give an economical formulation to the proof of
the isoperimetric theorem for curves on minimal surfaces.

Let $C$ be a smooth simple closed curve in Euclidean $n$-space, where $n \geq 2$.
In what follows we shall assume, as can be achieved by a translation, that the
centroid of arc length of $C$ is at the origin. Hence, if $x$ is the position vector,
we assume

$$\int_C x \, ds = 0. \quad (1)$$

If we let $y(t) = x(Lt/2\pi)$, $0 \leq t \leq 2\pi$, where $L$ is the length of $C$, then (1)
allows us to apply Wirtinger’s inequality [1, p. 105] componentwise to obtain

$$\int_0^{2\pi} |y|^2 \, dt \leq \int_0^{2\pi} \left| \frac{dy}{dt} \right|^2 \, dt, \quad (2)$$

from which follows

$$\int_C |x|^2 \, ds \leq \frac{L^2}{4\pi^2} \int_C \left| \frac{dx}{ds} \right|^2 \, ds = \frac{L^3}{4\pi^2}. \quad (3)$$

Denoting the integral on the far left-hand side of (3) by $I$, we have then an
inequality of Sachs [7],

$$L^3 - 4\pi^2 I \geq 0. \quad (4)$$

Now, with the same hypotheses as above, suppose $C$ is the boundary of a
smooth orientable surface $S$ with area $A$. Then we have the area formula (see
[3], [4], [5])

$$\int_C x \cdot v \, ds = 2A + 2 \iint_S x \cdot H \, dA, \quad (5)$$

where $v$ is the outward pointing unit normal to $C$ tangent to $S$, and $H$ is the
mean curvature vector of $S$, which satisfies $H \equiv 0$ in case $S$ is a minimal

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surface. In analogy to the procedure of Sachs [7] we form the vector field
\[ D = x - (L/2\pi\nu) \nu \] along \( C \) and define the functional
\[ R^2 = \frac{1}{L} \int_C D \cdot D \, ds = \frac{1}{L} \int_C x \cdot \nu \, ds + \frac{L^2}{4\pi^2}. \] (6)
Substitution from (5) into (6) and rearrangement of terms gives
\[ L^2 - 4\pi A = 2\pi^2 R^2 + \frac{1}{2L} (L^3 - 4\pi^2 I) + 4\pi \int_S x \cdot H \, dA. \] (7)
Applying (4), we have
\[ L^2 - 4\pi A \geq 2\pi^2 R^2 + 4\pi \int_S x \cdot H \, dA, \] (8)
giving, when \( H \equiv 0 \), a sharpened form of the isoperimetric inequality for curves on minimal surfaces (see [2]–[6]).

When \( S \) is a minimal surface, (7) yields
\[ 4\pi^2 I - 8\pi L A + L^3 = 4\pi^2 L R^2 \geq 0, \] (9)
which can be rearranged as
\[ L^2 - 4\pi A \geq (4\pi / L)(LA - \pi I). \] (10)
On the other hand, (4) can be written in the form
\[ L^2 - 4\pi A \geq (4\pi / L)(\pi I - LA). \] (11)
From (10) and (11) we have another sharpening of the isoperimetric inequality for curves on minimal surfaces,
\[ L^2 - 4\pi A \geq (4\pi / L)|\pi I - LA|, \] (12)
an inequality established in the same fashion for plane curves by Sachs [7].

REFERENCES


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