EQUIVALENCES GENERATED BY FAMILIES OF BOREL SETS

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Abstract. The equivalence relation on the reals generated by a family of \( \mathcal{B} \) Borel sets has either less than \( \aleph_0 \) or else exactly \( 2^{\aleph_0} \) equivalence classes.

As is usual in modern set theory, we identify an ordinal with the set of its predecessors, and a cardinal with the first ordinal of that cardinality. Thus \( 2 = \{0, 1\} \), \( \omega = \{0, 1, 2, \ldots\} \); while \( \aleph_0 = \omega \), \( \aleph_1 \) is the first uncountable ordinal \( \omega_1 \), etc. If \( \alpha, \beta \) are ordinals,

\[
\beta \alpha = \{ f : f \text{ is a function} \& \text{dom } f = \beta \& \text{range } f \subseteq \alpha \};
\]

while

\[
\beta \alpha = \bigcup_{\gamma < \beta} \gamma \alpha.
\]

If \( f \in \beta \alpha \) and \( \gamma < \beta \), then \( f|\gamma \) is the restriction of \( f \) to \( \gamma \); while if \( \delta < \alpha \), \( f \circ \delta \)

is the element \( g \) of \( (\beta + 1)\alpha \) with \( g|\beta = f \) and \( g(\beta) = \delta \).

Let \( X \) be an uncountable Polish space (separable topological space admitting a complete metric), e.g. the reals. A family \( \mathcal{S} \) of subsets of \( X \) generates an equivalence relation \( E(\mathcal{S}) \) on \( X \) defined by

\[
x E(\mathcal{S}) y \iff \forall S \in \mathcal{S} (x \in S \iff y \in S).
\]

Let \( k \) be an infinite cardinal. A subset \( S \subseteq X \) is called \( k \)-Souslin if \( S \) can be represented in the form

\[
S = \bigcup_{f \in \kappa} \bigcap_{n < \omega} C_y^n,
\]

where for each \( s \in \kappa \), \( C_s \subseteq X \) is closed. \( S \) is co-\( k \)-Souslin if \( X - S \) is \( k \)-Souslin, and bi-\( k \)-Souslin if both \( k \)-Souslin and co-\( k \)-Souslin. Thus the \( \omega \)-Souslin sets are just the analytic \( (\Sigma_1^1) \) sets; the co-\( \omega \)-Souslin sets are the \( \text{CA}(\Pi^0_1) \) sets; and by a classical theorem of Souslin (see [3]) the bi-\( \omega \)-Souslin sets are the Borel sets.

An equivalence relation \( E \) on \( X \) is said to have perfectly many classes if there is a perfect (closed, dense-in-itself) \( P \subseteq X \) such that no two (distinct) elements of \( P \) are \( E \)-equivalent. Since any perfect subset of \( X \) has cardinality \( 2^{\aleph_0} \), this implies \( E \) has \( 2^{\aleph_0} \) classes. Note that if \( S, T \) are families of subsets of
$X$ with $\mathcal{T} \subseteq \mathcal{S}$, then $E(\mathcal{T}) \subseteq E(\mathcal{S})$ (as subsets of $X^2$), and, hence, the number of $E(\mathcal{S})$ classes can be no less than the number of $E(\mathcal{T})$ classes, and the former has perfectly many classes if the latter does.

**Theorem.** Let $X$ be an uncountable Polish space, $\kappa$ an infinite cardinal, $\mathcal{S}$ a family of $\kappa$ many bi-$\kappa$-Souslin subsets of $X$. Then if the equivalence relation $E(\mathcal{S})$ generated by $\mathcal{S}$ has more than $\kappa$ equivalence classes, there exists a countable $\mathcal{T} \subseteq \mathcal{S}$ such that $E(\mathcal{T})$ has perfectly many classes.

**Proof.** Fix a complete metric $\rho$ on $X$ compatible with its topology. Enumerate $\mathcal{S} = \{S^\alpha : \alpha < \kappa\}$. For each $\alpha < \kappa$ fix families $\mathcal{C}_s^\alpha$ of closed subsets of $X$ for $i = 0, 1$ and $s \in \kappa$, such that:

$$S^\alpha = \bigcup_{f \in \kappa} \bigcap_{n \in \omega}^0 \mathcal{C}_{f|n}^\alpha, \quad X - S^\alpha = \bigcup_{f \in \kappa} \bigcap_{n \in \omega}^1 \mathcal{C}_{f|n}^\alpha.$$ 

We may choose these families to be nested, so $s \subseteq t$ implies $\mathcal{C}_s^\alpha \subseteq \mathcal{C}_t^\alpha$, and we may choose them so that for $n \in \omega$ and $s \in \kappa$, the $\rho$-diameter of $\mathcal{C}_s^\alpha$ is less than $2^{-n}$. For $\alpha < \kappa$, $i = 0, 1$, and $s \in \kappa$, set

$$\mathcal{S}_s^\alpha = \bigcup_{f \in \kappa} \bigcap_{n \in \omega}^i \mathcal{C}_{f|n}^\alpha \subseteq \mathcal{C}_s^\alpha.$$ 

Assume $E(\mathcal{S})$ has $> \kappa$ classes, and let $Z \subseteq X$ be a set of $\kappa$-many $\mathcal{S}$-inequivalent elements.

We will define for every $l \in \omega$, $\sigma \in \omega^2$, an ordinal $\alpha(\sigma) < \kappa$ and elements $s(\sigma, k)$ of $\kappa^\omega$ for $k < l$, so that setting

$$(1) \quad T_\sigma = \bigcap_{k < l} \alpha(\sigma, k) S_{(\sigma, k)}^\alpha \subseteq \bigcap_{k < l} \alpha(\sigma, k) C_{(\sigma, k)}^\alpha,$$

we have $\text{card}(Z \cap T_\sigma) = \kappa^+$. We will also arrange matters so that $\sigma \subseteq \tau$ implies $s(\sigma, k) \subseteq s(\tau, k)$ for all relevant $k$. We proceed by induction. Suppose then that $l \in \omega$, $\sigma \in \omega^2$, and suppose that for all $k < l$, $\alpha(\sigma|k)$ and $s(\sigma, k)$ have been defined and satisfy the conditions above.

In particular, $\text{card}(Z \cap T_\sigma) = \kappa^+$. We claim this assumption implies that there exists an $\alpha < \kappa$ such that both $Z \cap T_\sigma \cap S^\alpha$ and $(Z \cap T_\sigma) - S^\alpha$ have cardinality $\kappa^+$. For suppose the opposite, and setting, for each $\alpha < \kappa$, $M^\alpha$ is whichever of $Z \cap T_\sigma \cap S^\alpha$ or $(Z \cap T_\sigma) - S^\alpha$ has cardinality $< \kappa$, we would find that all elements of $Z - \bigcup_{\alpha < \kappa} M^\alpha$ would be $E(\mathcal{S})$-equivalent, hence that there could be only one such element, hence that $\text{card} Z = \kappa$, a contradiction! Let $\alpha(\sigma)$ be the least $\alpha$ with $\text{card}(Z \cap T_\sigma \cap S^\alpha) = \text{card}((Z \cap T_\sigma) - S^\alpha) = \kappa^+$. Now $Z \cap T_\sigma \cap S^{\alpha(\sigma)}$ is contained in

$$\bigcap_{k < l} \alpha(\sigma|k) S_{(\sigma, k)}^\alpha \cap S^{\alpha(\sigma)} = \bigcap_{k < l} \bigcup_{\tau < \kappa}^{\alpha(\sigma|k)} S_{(\sigma, k|\tau)}^{\alpha(\sigma|k)} \cap \bigcup_{s \in \omega^{l+1}}^{\alpha(\sigma)} S_s^{\alpha(\sigma)}.$$ 

So there exist $\nu_0, \nu_1, \ldots, \nu_{(l-1)}$ and $s$ such that setting $s(\sigma \ast 0, k) = s(\sigma, k) \ast \nu_k$ for $k < l$, and $s(\sigma \ast 0, l) = s$, and defining $T_{\sigma \ast 0}$ as per (1) above,
we still have \( \text{card}(Z \cap T_{\sigma,0}) = \kappa^+ \). The \( s(\sigma \cdot 1, k) \) for \( k < l \) are similarly defined.

For \( g \in \omega^\omega \), \( \{ T_{g,n} : n \in \omega \} \) forms a nested sequence of nonempty closed sets with \( \rho \)-diameters converging to 0. Hence this family intersects in a point \( x_g \in X \). If \( g(m) = 0 \), then \( x_g \) belongs to

\[
\bigcap_{n > m} \mathcal{C}_k^{\alpha(g[m])} \subseteq S^{\alpha(g[m])}.
\]

Similarly, if \( g(m) = 1 \), then \( x_g \notin S^{\alpha(g[m])} \). Thus if \( g, h \) are two (distinct) elements of \( \omega^\omega \), \( x_g, x_h \) are \( E(S) \)-inequivalent, and incidentally \( x_g \neq x_h \). Thus \( A = \bigcup_{g \in \omega^\omega} \bigcap_{n \in \omega} T_{g,n} = \{ x_g : g \in \omega^\omega \} \) is an uncountable analytic set, and hence contains a perfect subset \( P \). Moreover, setting \( \mathcal{S} = \{ S^\alpha(a) : \alpha \in 2 \} \), any two elements of \( P \) are \( E(\mathcal{S}) \)-inequivalent, proving the theorem. □

**Corollary 1.** Let \( X \) be a Polish space, \( \kappa \) an infinite cardinal. Then any equivalence relation on \( X \) which is an intersection of \( \kappa \) CA equivalences has either \( < \kappa \) or else perfectly many equivalence classes.

**Proof.** We use a deep theorem of Silver [4]: Any CA(\( \Pi^1_1 \)) equivalence relation on a Polish space \( X \) has either countably many or else perfectly many equivalence classes. Now let \( E \) be an equivalence on a Polish space \( X \) of form \( \bigcap_{\alpha < \kappa} E_\alpha \) where the \( E_\alpha \) are CA equivalences. If any \( E_\alpha \) has perfectly many classes, so does \( E \). If each \( E_\alpha \) has only countably many classes \( \{ S_{\alpha,n} : n < N_\alpha \} \), \( N_\alpha < \omega \), then each of these \( S_{\alpha,n} \) is both CA (since \( E_\alpha \) is CA) and analytic (being the complement of \( \bigcup_{n \neq n'} S_{\alpha,m} \)) and hence is Borel. Thus in this case \( E = E(S) \) where \( S = \{ S_{\alpha,n} : \alpha < \kappa, n < N_\alpha \} \) is a family of \( \kappa \) Borel sets. Thus any intersection of \( \kappa \) CA equivalences either has perfectly many classes or else is generated by a family of \( \kappa \) Borel sets. Corollary 1 is immediate. This corollary answers a question of J. Steel. □

The referee has informed us that V. Harnik and M. Makkai [5] have obtained Corollary 1 (for \( X = \) Baire space) by a model-theoretic argument. The Theorem has somewhat more scope than this corollary, implying e.g. that if \( \mathcal{S} \) is a family of \( \aleph_1 \) analytic sets, \( E(\mathcal{S}) \) has \( < \aleph_1 \) or perfectly many classes.

**Corollary 2.** Any analytic equivalence relation on a Polish space \( X \) has either \( < \omega_1 \) or else perfectly many classes.

**Proof.** Elsewhere [2] we have shown: Any analytic equivalence relation on a Polish space \( X \) is an intersection of \( \omega_1 \) Borel equivalences. Corollary 2 is then immediate. Actually in [2] we establish more: A CPC4(\( \Pi^1_1 \)) equivalence of the special form \( xEy \leftrightarrow \forall z \in X (x, y, z) \in D \), where \( D \subseteq X^3 \) is analytic, and for each fixed \( z \), \( \{ (x, y) : (x, y, z) \in D \} \) is an equivalence relation, is an intersection of \( \omega_1 \) CA equivalences. So the cardinal estimates on the number of classes in Corollary 2 apply to such special CPC4 equivalences, too. Corollary 2 was the main result of our thesis [1]. It answers a question of H. Friedman. □
BIBLIOGRAPHY


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