THE $SC_{k+1}P$-INTEGRAL AND TRIGONOMETRIC SERIES

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ABSTRACT. Recently P. S. Bullen and C. M. Lee defined a scale of symmetric Cesàro-Perron integrals, but left open the question of whether their $SC_{k+1}P$-integral solves the coefficient problem for $(C, k)$ summable series. This paper gives an affirmative answer to that question under natural conditions.

1. Introduction. The problem of constructing an integral in terms of which the coefficients of a $(C, k)$ summable trigonometric series may be represented has been solved in several ways (see, e.g., [2], [6]-[10], [12], and [13]). In the case $k = 0$, J. C. Burkill's SCP-integral [6] solves the representation problem, although the integration by parts formula which was used in the original solution has yet to be verified. Recently H. Burkill [4] obtained the result without recourse to an integration by parts formula, using instead convergence properties of a series which is the formal product of a trigonometrical series and $\cos px$ or $\sin px$.

Bullen and Lee [3] have introduced a scale of symmetric Cesàro-Perron integrals, but left open the question of whether their $SC_{k+1}P$-integral solves the coefficient problem for $(C, k)$-summable series.

This paper gives an affirmative answer to that question under certain assumptions.

Our main result (Theorem 1) is weaker than the result in James [12, Theorem 6.2], since it requires the $(C, k)$summability of both the trigonometrical series and its conjugate except on a countable set. The result is weaker too than that of Mukhopadhyay [13, Theorem 8.1] for the same reason and also because in the latter the original series is required only to be summable $(C, k)$ a.e. and to have bounded Cesàro means except on a countable set.

Insofar as application to trigonometrical series is concerned, the $SC_{k+1}P$-integral is thus seen to have the power of an “unsymmetric” rather than a “symmetric”, integral (cf. [6] and [8]). This is related to the fact that in the formulation of the $SC_{k+1}P$-integral, the definition of $SC_{k+1}DF(x)$ is given in terms of the $C_kP$-integral (rather than the $SC_kP$-integral) of $F(x)$ because of the absence of an integration by parts formula for the $SCP$-integral.

2. Definitions. The notation and theory of the $C_kP$-integral [5] is assumed. For the theory of the $SC_kP$-integral we refer the reader to [3], but for
convenience we reproduce the definition here.

Let $F$ be a $C_{k-1}P$-integrable function on $[a, b]$, $k > 1$, and define
\[
\Delta_k(F; x, h) = \frac{k + 1}{2h} \{ C_k(F; x, x + h) - C_k(F; x, x - h) \},
\]
and
\[
SC_k^D F(x) = \liminf_{h \to 0} \Delta_k(F; x, h),
\]
where $x \in (a, b)$ and $C_k(F; x, x + h)$ is the $k$th Cesàro mean of $f$ in $(x, x + h)$ as defined in [5]. Define $SC_k^D F$ and $SC_k D F$ in the usual way. The function $F$ is said to be $SC_k$-continuous at $x$ if $\lim_{h \to 0+} h \Delta_k(F; x, h) = 0$. It is easily seen that $F$ is $SC_k$-continuous at $x$ whenever it is $C_k$-continuous at $x$, and that $SC_k D F(x)$ exists and equals $C_k D F(x)$ whenever the latter exists.

Now suppose that $f$ is a function defined and finite almost everywhere in $[a, b]$ and that $B$ is a subset of $[a, b]$ of measure $b - a$, $a, b \in B$. The $C_{k-1}P$-integrable functions $M$ and $m$ are $SC_kP$-major and minor functions, respectively, of $f$ on $[a, b]$ with base $B$ if:

1. $M$ and $m$ are $SC_kP$-continuous on $[a, b]$ and $C_k$-continuous on $B$;
2. $SC_k^D M(x) > f(x) > SC_k^D m(x)$, a.e. in $(a, b)$;
3. $SC_k^D M(x) > -\infty$ and $SC_k^D m(x) < +\infty$ except perhaps in a scattered set;
4. $M(a) = 0 = m(a)$.

If $f$ has $SC_kP$-major and minor functions and if
\[
I \equiv \inf M(b) = \sup m(b) \neq \pm \infty,
\]
then $f$ is $SC_kP$-integrable on $[a, b]$ with base $B$ and we write
\[
I = SC_kP \int_{[a, b]} f(t) \, dt.
\]

Suppose that $F$ is a function defined on the interval $[a, b]$. If for $x_0 \in [a, b]$ there exist constants $\alpha_1, \alpha_2, \ldots, \alpha_r$ which depend on $x_0$ only and not on $h$, such that
\[
F(x_0 + h) - F(x_0) = \sum_{k=1}^{r} \alpha_k \frac{h^k}{k!} + o(h^\tau), \quad \text{as } h \to 0,
\]
then $\alpha_k$, $1 \leq k \leq r$, is called the Peano derivative of order $k$ of $F$ at $x_0$ and is denoted by $F_{(k)}(x_0)$. If there exist constants $\beta_0, \beta_2, \ldots, \beta_{2r}$ which depend on $x_0$ only and not on $h$, such that
\[
\frac{F(x_0 + h) + F(x_0 - h)}{2} = \sum_{k=0}^{r} \beta_{2k} \frac{h^{2k}}{(2k)!} + o(h^{2r}), \quad \text{as } h \to 0,
\]
then $\beta_{2k}$, $0 \leq k \leq r$, is called the de la Vallée Poussin derivative of order $2k$ of $F$ at $x_0$ and is denoted by $D^{2k} F(x_0)$. If $F$ has derivatives $D^{2k} F(x_0)$, $0 \leq k \leq r - 1$, we write
\[
\frac{h^{2r}}{(2r)!} \theta_{2r}(F; x_0, h) = \frac{F(x_0 + h) + F(x_0 - h)}{2} - \sum_{k=0}^{r-1} \frac{h^{2k}}{(2k)!} D^{2k} F(x_0),
\]
and define
\[ \overline{D}^{2r}F(x_0) = \lim_{h \to 0} \sup \theta_{2r}(F; x_0, h) \]
and
\[ \overline{D}^{2r}F(x_0) = \lim_{h \to 0} \inf \theta_{2r}(F; x_0, h). \]
The de la Vallée Poussin derivatives are defined similarly for odd-numbered indices (see, e.g., [11, pp. 163–164]).

If \( F(x_0) \) exists, so does \( D^{r}F(x_0) \) and \( F(x_0) = D^{r}F(x_0) \).

3. The representation of trigonometrical series in Fourier form. The series
\[ \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \left( a_n \cos nx + b_n \sin nx \right) = \sum_{n=1}^{\infty} \frac{a_n(x)}{2} \]
is said to satisfy condition A if

(i) \( 3.1.1 \) and the series conjugate to \( 3.1.1 \), namely
\[ \sum (b_n \cos nx - a_n \sin nx) = \sum b_n(x), \]
are both summable \((C, k)\) on \([0, 2\pi] - E \equiv B\), where \( E \) is a countable subset of \([0, 2\pi]\), and

(ii) \( A_{k-1}^{x}(x) = o(n^{k}) \), \( B_{k-1}^{x}(x) = o(n^{k}) \) for \( x \in E \), where \( A_{k-1}^{x}(x) \) and \( B_{k-1}^{x}(x) \) denote the Cesàro means of order \( k - 1 \) of series \( 3.1.1 \) and \( 3.2 \), respectively, at \( x \).

Clearly (i) implies \( a_n = o(n^{k}) \), \( b_n = o(n^{k}) \), \( A_{k-1}^{x}(x) = o(n^{k}) \) and \( B_{k-1}^{x}(x) = o(n^{k}) \), \( x \in [0, 2\pi] - E \).

If \( 3.1.1 \) satisfies condition A we consider the following array of series:
\[ \frac{a_0}{2} + \sum a_n(x) = f(x), \quad (C, k), \quad x \in [0, 2\pi] - E, \]
\[ \frac{a_0x}{2} - \sum \frac{b_n(x)}{n} = F^{k+1}(x), \quad (C, k - 1), \]
\[ x \in [0, 2\pi] - E, \]
\[ \frac{a_0x^2}{2 \cdot 2} - \sum \frac{a_n(x)}{n^2} = F^k(x), \quad (C, k - 2), \]
\[ x \in [0, 2\pi] - E, \]
where the convergence of series \( 3.1.(k + 3) \) is uniform to a continuous function \( F \).

It follows from Theorem 3.1 [11] that \( F \) is \((k + 2)\) smooth everywhere, and, at points of \( B \),
\[ \frac{a_0x^{2r}}{2 \cdot (2r)!} + (-1)^r \sum_{n=1}^{\infty} \frac{a_n(x)}{n^{2r}} = D^{k+2-2r}F(x), \quad (C, k - 2r), \]
for \( 0 \leq r \leq ((k + 1)/2) \). Also by Theorem 3.2 [12] \( F_{r}(x) \) exists for \( 0 \leq r \leq k \) and \( x \in [0, 2\pi] \) (and equals \( D_{r}F(x) \)).
Since $F(x)$ can also be obtained by integrating (3.1.2) formally term by term $(k + 1)$ times, it follows from Theorem 3.1 [12] again that $D^{k+1}F(x)$ exists and equals $F^{k+1}(x)$ on $B$. It follows easily that $D^{k+1}F(x) = F^{(k+1)}(x)$ on $B$. It may be shown similarly that $F^{(r)}(x) = F^{(r)}(x)$, $x \in B$, $2 \leq r \leq k$. Finally it follows from a lemma of S. Verblunsky [14, Lemma 5] that $F^{(0)}(x) = F^{(0)}(x)$.

**Lemma 1.** If (3.1.1) satisfies condition $A$ then $F^{r+1}(x)$ is $C_r$ integrable on $[0, 2\pi]$ and

$$F^{(r)}(t)\bigg|_0^x = C_r \int_0^x F^{r+1}(t) \, dt = C_r \int_0^x F^{(r+1)}(t) \, dt,$$

$0 < r < k, 0 < x < 2\pi$, where $F^{(0)}(x) = F(x)$.

**Proof.** We have for each $r, 0 < r < k$, $F^{(r)}(x)$ exists everywhere in $[0, 2\pi]$ and $F^{(r+1)}(x) = F^{(r+1)}(x)$ except possibly on a countable set. (3.3) follows from [1, Propositions 4.10 and 4.11].

**Lemma 2.** If (3.1.1) satisfies condition $A$, then $F^{k+1}(x)$ is $C_{k+1}$-continuous for $x \in B$.

**Proof.** By $(k - 1)$ integrations by parts (cf. [11, p. 169]) we obtain

$$
\frac{(k + 1)!}{h^{k+1}} \left( \frac{1}{k!} \right) C_k \int_x^{x+h} (x + h - t)^k F^{k+1}(t) \, dt
\]

$$
= \frac{(k + 1)!}{h^{k+1}} \left( \frac{1}{k!} \right) C_k \int_x^{x+h} (x + h - t)^k F^{(k+1)}(t) \, dt
\]

$$
= \frac{(k + 1)!}{n^{k+1}} \left[ F(x + h) - F(x) - \sum_{i=1}^{k} \frac{h^i}{i!} F^{(i)}(x) \right]
\]

$$
\rightarrow F^{(k+1)}(x) = F^{k+1}(x), \quad \text{for } x \in B,
\]

by assuming, without loss of generality, that $F^{(r)}(0) = 0, 0 < r < k$.

**Lemma 3.** If (3.1.1) satisfies condition $A$ then $f(x)$ is $SC_{k+1}$ integrable with base $B$ on $[\alpha, \alpha + 2\pi], \alpha \in B$, and

$$F^{k+1}(x)\bigg|_\alpha^{\alpha+2\pi} = SC_{k+1} \int_{[\alpha,\alpha+2\pi]} f(t) \, dt.$$ 

**Proof.** We have by Lemma 2 that $F^{k+1}(x)$ is $C_{k+1}$-continuous in $B$. Since $F(x)$ is $(k + 2)$-smooth everywhere in $(0, 2\pi)$ [12, Theorem 3.1] and $D_{k-2i}F(x) = F^{(k-2i)}(x), 0 < i < [k/2]$, it follows [3, Remark, p. 1279], that $F$ is $SC_{k+1}$-continuous everywhere in $(0, 2\pi)$.

Moreover, since

$$C_{k+1}(F^{k+1}; x, x + h) - C_{k+1}(F^{k+1}; x, x - h)
\]

$$
= \frac{2}{k + 2} h\theta_{k+2}(F; x, h),
\]

it is clear that
SC\textsubscript{k+1}DF\textsuperscript{k+1}(x) = \lim_{h \to 0} \theta_{k+2}(F; x, h) = D_{k+2}F(x) = f(x), \quad x \in B.

Since it is known [12, Theorem 5.1] that the set of points for which either
\[ \lim_{h \to 0} \theta_{k+2}(F; x, h) = -\infty \quad \text{or} \quad \lim_{h \to 0} \theta_{k+2}(F; x, h) = +\infty \]
is a scattered set, it follows from (3.5) that the set of points for which either
\[ SC\textsubscript{k+1}D \ast F\textsuperscript{k+1}(x) = -\infty \quad \text{or} \quad SC\textsubscript{k+1}D \ast F\textsuperscript{k+1}(x) = +\infty \]
is a scattered set.

We see then that \( F\textsuperscript{k+1}(x) - F\textsuperscript{k+1}(\alpha) \) is both an \( SC\textsubscript{k+1}P \text{-major} \) and an \( SC\textsubscript{k+1}P \text{-minor} \) function of \( f \) on \([a, b]\) with base \( B \) and (3.4) holds.

If (3.1.1) satisfies condition A then its formal product with \( \cos px \), \( p = 1, 2, \ldots \),
\[ \frac{1}{2} u_o + \sum_{n=1}^{\infty} \left( u_n \cos nx + v_n \sin nx \right) \equiv \sum_{n=0}^{\infty} u_n(x), \quad (3.6) \]
and the formal product of (3.2) with \( \cos px \),
\[ \sum_{n=1}^{\infty} \left( +v_n \cos nx - u_n \sin nx \right) \equiv -\sum_{n=0}^{\infty} v_n(x) \quad (3.7) \]
are summable \( (C, k) \) in \( B \), the former to sum \( f(x) \cos px \) [12]. Consequently, the series obtained by integrating (3.6) formally term-by-term,
\[ \frac{1}{2} u_0 x + \sum_{n=1}^{\infty} \frac{v_n(x)}{n}, \quad (3.8) \]
is summable \( (C, k - 1) \) in \( B \). In the following we denote the \( (C, k - 1) \) sum of (3.8) in \( B \) by \( G\textsuperscript{k+1}(x) \). Similarly the sum of the series obtained by forming the product of (3.6) with \( \sin px \) and integrating term-by-term will be denoted by \( H\textsuperscript{k+1}(x) \).

Now using the same methods as in Lemma 3 we obtain

**Lemma 4.** If (3.1.1) satisfies condition A, then for \( \alpha \in B \), \( f(x)\cos px \) and \( f(x)\sin px \), \( p = 1, 2, \ldots \), are \( SC\textsubscript{k+1}P \text{-integrable} \) on \([a, a + 2\pi]\) for \( p = 1, 2, \ldots \) with base \( B \). Moreover,
\[ G\textsuperscript{k+1}(x)|_{a}^{a+2\pi} = SC\textsubscript{k+1}P \int_{[a\alpha + 2\pi]}^{B} f(t) \cos pt \, dt, \]
and
\[ H\textsuperscript{k+1}(x)|_{a}^{a+2\pi} = SC\textsubscript{k+1}P \int_{[a\alpha + 2\pi]}^{B} f(t) \sin pt \, dt. \]

**Theorem 1.** If (3.1.1) satisfies condition A then
\[ a_p = \frac{1}{\pi} \int_{[a\alpha + 2\pi]}^{B} f(t) \cos pt \, dt, \quad p = 0, 1, 2, \ldots, \]
and
\[ b_p = \frac{1}{\pi} \int_{[a\alpha + 2\pi]}^{B} f(t) \sin pt \, dt, \quad p = 1, 2, \ldots. \]
PROOF. In the case of $a_0$ we have, by Lemma 3, 
\[
\frac{a_0}{2} \int_{a}^{a+2\pi} = a_0 \pi \ = \ SC_{k+1} \int_{[a,a+2\pi]} f(t) \cos nt \ dt.
\]
The result for $a_p$ follows in analogous fashion from Lemma 4 since $\frac{1}{2} u_0 = \frac{1}{2} a_p$, and the result for $b_p$ is obtained in an exactly similar way.

REFERENCES


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