DOUBLE COMMUTANTS OF $C_0$ CONTRACTIONS

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Abstract. D. Sarason has shown that an operator in the double commutant of a contraction $T$ of class $C_0(1)$ is interpolated by a function in $H^\infty$, that is, $(T)' = \{\phi(T); \phi \in H^\infty\}$, [3]. Generally, in [4] Sz.-Nagy and C. Foiaş have shown that an operator in the double commutant of a contraction $T$ of class $C_0(n)$ is interpolated by a function in $N_T$, that is, $(T)' = \{\phi(T)\phi(T); \phi_1/\phi_i \in N_T\}$. In this note we shall show that an operator in the double commutant of a $C_0$ contraction $T$ with finite defect indices $\delta_T > 0$ is interpolated by a function in $H^\infty$.

1. Introduction. We begin by recalling notations of [6]. If $T$ is a contraction on a separable Hilbert space $H$ such that $T'' \to 0$ (strongly) as $n \to \infty$, then $T$ is said to be of class $C_0$. If $T$ and $T^*$ are of class $C_0$, then $T$ is said to be of class $C_0$. For a contraction $T$, $\delta_T = \text{rank}(1 - T^*T)$ and $\delta_T^* = \text{rank}(1 - TT^*)$ are called defect indices of $T$. If $T$ is of class $C_0$ and $\delta_T = \delta_T^* = n < \infty$, then $T$ is said to be of class $C_0(n)$. If $T$ is of class $C_0$ and $\delta_T = \delta_T^* = n < \infty$, then $T$ is of class $C_0(n)$.

Let $\Theta$ be an $n \times m$ matrix over the Hardy space $H^\infty$ of bounded measurable functions on the unit circle with vanishing Fourier coefficients of negative indices. Such a matrix is called inner if $\Theta^*(\lambda)\Theta(\lambda) = 1_m$ a.e. on the unit circle. In this case it necessarily follows that $n \geq m$. For $T$ of class $C_0$ with finite defect indices $\delta_T = n$ and $\delta_T^* = m$, there exists an $n \times m$ inner function $\Theta$ such that $T$ is unitarily equivalent to $S(\Theta)$ on $H(\Theta)$, which are defined by next relations:

$$H(\Theta) = H_n^2 \Theta H_m^2$$

and

$$S(\Theta)h = P_\Theta SH \text{ for } h \in H(\Theta),$$

where $H_n^2$ is the Hardy space of $n$-dimensional column vector valued functions on the unit circle, $P_\Theta$ is the orthogonal projection of $H_n^2$ onto $H(\Theta)$, and $(Sh)(\lambda) = \lambda h(\lambda)$.

An $n \times m$ ($n \geq m$) normal inner matrix $N'$ over $H^\infty$ is, by definition, of the form:

Received by the editors September 14, 1976 and, in revised form, December 28, 1976 and April 25, 1977.


Key words and phrases. $C_0$ contraction, double commutant, inner matrix, lifting theorem.

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where for each $i$, $\nu_i$ is a scalar inner function and a divisor of its successor.

Let $\Theta$ be an $n \times m$ ($\infty > n > m$) inner matrix over $H^\infty$ and $N$ a "corresponding" normal matrix i.e., $N$ is an $n \times m$ normal inner matrix of the form (1), where each entry $\nu_i$ is the invariant factor of $\Theta$, that is,

$$\nu_i = \frac{d_i}{d_{i-1}} \text{ for } i = 1, 2, \ldots, m,$$

where $d_0 = 1$ and $d_i$ is the largest common inner divisor of the minors of order $i$. Then Nordgren [2] has shown that there exist $n \times n$ matrices $\Delta$, $\Delta_k$ and $m \times m$ matrices $\Lambda$, $\Lambda_k$ ($k = 1, 2$) satisfying

$$\Delta \Theta = N \Lambda, \quad \Delta_k N = \Theta \Lambda_k,$$

$$(\det \Delta)(\det \Lambda) \land d_m = 1, \quad (\det \Delta_k)(\det \Lambda_k) \land d_m = 1,$$

$$(\det \Delta_1) \land (\det \Delta_2) = 1,$$

where $x \land y$ denotes the largest common inner divisor of the scalar functions $x$ and $y$ in $H^\infty$. Setting $X = P_N \Delta|H(\Theta)$ and $Y_k = P_\Theta \Delta_k|H(N)$, one obtains one-to-one operators $X$, $Y_1$ and $Y_2$ such that

$$XS(\Theta) = S(N)X, \quad Y_k S(N) = S(\Theta)Y_k \text{ and}$$

$$Y_1 H(N) \lor Y_2 H(N) = H(\Theta),$$

where $L_1 \lor L_2$ denotes the minimum subspace including both $L_1$ and $L_2$ (see [5]).

For a completely nonunitary contraction $T$, it is possible to define $\phi(T)$ for every function $\phi$ in $H^\infty$. In particular, for $S(\Theta)$ given above, $\phi(S(\Theta))$ can be equivalently defined by the following:

$$\phi(S(\Theta))h = P_{\Theta} \phi h \text{ for } h \in H(\Theta) \text{ (see [3], [6]).}$$

The purpose of this note is to prove

**Theorem.** Let $\Theta$ be an $n \times m$ inner matrix over $H^\infty$ ($n > m$). Then

$$\{ S(\Theta) \}'' = \{ \phi(S(\Theta)): \phi \in H^\infty \}.$$

2. **Proof of the theorem.** The following lemma is well known and called the lifting theorem.

**Lemma 1** ([3], [6]). Let $\Theta$ be an $n \times m$ inner matrix over $H^\infty$. Then for an operator $T$ commuting with $S(\Theta)$ there is an $n \times n$ matrix $\Psi$ over $H^\infty$ such that

$$
The following lemma is analogous to Lemma 1 of [7] and the proof is simple. Thus we omit it.

**Lemma 2.** Let \( N' \) be an \( n \times m \) normal inner matrix of the form (1). In order that an \( n \times n \) matrix \( \Psi = (\psi_{ij}) \) over \( H^\infty \) satisfies \( \Psi N' H^2_m \subseteq N' H^2_m \), it is necessary and sufficient that
1. \( \nu_i / \nu_j \) is a divisor of \( \psi_{ij} \) for \( 1 < j < i < m \),
2. \( \psi_{ij} = 0 \) for \( m + 1 < i < n \) and \( 1 < j < m \).

**Lemma 3.** If \( T \) belongs to \( \{ S(\Theta) \}'' \), then there exist \( \phi_1 \) and \( \phi_2 \) in \( H^\infty \) such that
\[
\phi_2(S(\Theta)) = \phi_1(S(\Theta)) T \quad \text{and} \quad \phi_1 \wedge d_m = 1. \tag{5}
\]

**Proof.** Let \( N \) be an \( n \times m \) normal inner matrix of the form (1) corresponding to \( \Theta \). Let us consider \( X, Y_1 \) and \( Y_2 \) defined above. Then we have
\[
Y_k A X \in \{ S(\Theta) \}' \quad \text{for every} \quad A \in \{ S(N) \}',
\]
and hence \( (Y_k A X) = T^k (Y_k A X) \) \( (k = 1, 2) \). Thus, on setting \( B_k = X Y_k \) and \( C_k = X T Y_k \), one obtains
\[
B_k A C_k = C_k A B_k \quad (k = 1, 2). \tag{6}
\]
By Lemma 1, \( T \in \{ S(\Theta) \}' \) and \( A \in \{ S(N) \}' \) imply that there are \( n \times n \) matrices \( \Gamma \) and \( \Psi \) such that
\[
\Gamma H^2_m \subseteq \Theta H^2_m, \quad T = P_\Theta \Gamma | H(\Theta),
\]
\[
\Psi NH^2_m \subseteq NH^2_m \quad \text{and} \quad A = P_N \Psi | H(N).
\]
Then it is obvious that \( B_k = P_N \Delta \Delta_k | H(N) \) and \( C_k = P_N \Delta \Gamma \Delta_k | H(N) \). Now let \( \psi_{ij}, b_{ij}^k \) and \( c_{ij}^k \) be the \((i,j)\)th entry of \( \Psi, \Delta \Delta_k \) and \( \Delta \Gamma \Delta_k \), respectively. Since (6) implies that
\[
\left\{ (\Delta \Gamma \Delta_k) \Psi(\Delta \Gamma \Delta_k) - (\Delta \Gamma \Delta_k) \Psi(\Delta \Delta_k) \right\} H^2_n \subseteq NH^2_m,
\]
it follows that for \( 1 < i, j < n \)
\[
\sum_{h,l=1}^n \left\{ b_{hl}^k \psi_{hi} c_{lj}^k - c_{lh}^k \psi_{hl} b_{lj}^k \right\} \in \nu_i H^\infty, \tag{7}
\]
where \( \nu_{m+1} = \cdots = \nu_n = 0 \).
Set \( \psi_{ij} = 1 \) for \((i,j) = (1, n)\) and \( \psi_{ij} = 0 \) for \((i,j) \neq (1,n)\). Then, since \( \psi_{ij} \) \((1 < i, j < n)\) satisfy the conditions (i) and (ii), by (7) we have
\[
b_{hi}^k c_{nj}^k - c_{hi}^k b_{nj}^k \in \nu_i H^\infty \quad \text{for} \quad 1 < i, j < n.
\]
Similarly, we can deduce that
\[
b_{ij}^k c_{ij}^k - c_{ij}^k b_{ij}^k \in \nu_i H^\infty \quad \text{for} \quad 1 < i, j, r < n. \tag{8}
\]
Thus it is clear that \((c_{nj}^k \Delta \Delta_k - b_{nj}^k \Delta \Gamma \Delta_k) H^2_n \subseteq NH^2_m\), which implies that
\[
c_{nj}^k(S(N)) B_k - b_{nj}^k(S(N)) C_k = 0. \tag{9}
\]
From (3) and (9) it follows that
\[ Xc_{n}^{k}(S(\Theta))Y_{k} = XY_{i}c_{n}^{k}(S(N)) = B_{k}c_{n}^{k}(S(N)) \]
\[ = b_{n}^{k}(S(N))C_{k} = b_{n}^{k}(S(N))XY_{k} = Xb_{n}^{k}(S(\Theta))TY_{k}. \]

Since \( X \) is one-to-one, we have
\[ c_{n}^{k}(S(\Theta))Y_{k} = b_{n}^{k}(S(\Theta))TY_{k}. \]  
(10)

(Remark. The above method is analogous to that in [1] or [4].) Let \( \delta_{ij}^{k} \) be the \((i,j)\)th cofactor of \( \Delta_{k} \). Then from (10) it follows that

\[ \left( \sum_{j=1}^{n} \delta_{ij}^{k}c_{nj} \right)(S(\Theta))Y_{k} = (\det \Delta_{k})(S(\Theta))TY_{k}. \]  
(11)

Let \( \delta_{ij}, \delta_{ij}^{k} \) and \( \gamma_{ij} \) be the \((i,j)\)th entries of \( \Delta, \Delta_{k} \) and \( \Gamma \), respectively. Then since
\[ b_{ij}^{k} = \sum_{l=1}^{n} \delta_{il}^{k} \delta_{lj}^{k} \quad \text{and} \quad c_{ij}^{k} = \sum_{h,l=1}^{n} \delta_{lh}^{k} \gamma_{hl} \delta_{lj}^{k}, \]
we have
\[
\begin{bmatrix}
  b_{11}^{k}, & \ldots, & b_{1n}^{k} \\
  \vdots & \ddots & \vdots \\
  b_{n-11}^{k}, & \ldots, & b_{n-1n}^{k} \\
  c_{n1}^{k}, & \ldots, & c_{nn}^{k}
\end{bmatrix}
= \Pi\Delta_{k}, \quad \text{where} \quad \Pi =
\begin{bmatrix}
  \delta_{11}, & \ldots, & \delta_{1n} \\
  \vdots & \ddots & \vdots \\
  \delta_{n-11}, & \ldots, & \delta_{n-1n} \\
  \sum_{h} \delta_{nh} \gamma_{h1}, & \ldots, & \sum_{h} \delta_{nh} \gamma_{hn}
\end{bmatrix}.
\]

Thus from (11) it follows that
\[ (\det \Delta_{k})(S(\Theta)) \cdot (\det \Pi)(S(\Theta))Y_{k} = (\det \Delta_{k})(S(\Theta)) \cdot (\det \Delta)(S(\Theta))TY_{k}. \]

The second equation of (2) implies that \( (\det \Delta_{k})(S(\Theta)) \) is one-to-one (see [8]). Hence it follows that

\[ (\det \Pi)(S(\Theta))Y_{k} = (\det \Delta)(S(\Theta))TY_{k}. \]

From this and (4) it is clear that
\[ (\det \Pi)(S(\Theta)) = (\det \Delta)(S(\Theta))T. \]

Consequently, if \( \phi_{1} = \det \Delta \) and \( \phi_{2} = \det \Pi \), then this and the first equation of (2) imply that Lemma 3 is true.

**Proof of Theorem.** \( \phi_{2}(S(\Theta)) = \phi_{1}(S(\Theta))T \) implies that there exists an \( m \times n \) matrix \( \Omega = (\omega_{ij}) \) over \( H^{\infty} \) such that \( \phi_{2} - \phi_{1} \Gamma = \Theta \Omega \). Setting \( \theta_{ij} \) the \((i,j)\)th entry of \( \Theta \), we have
\[ - \phi_{1} \gamma_{ij} = \sum_{k=1}^{m} \theta_{ik} \omega_{kj} \quad \text{for} \quad i \neq j \]  
(12)
and
\[ \phi_{2} - \phi_{1} \gamma_{ii} = \sum_{k=1}^{m} \theta_{ik} \omega_{ki} \quad \text{for} \quad i = 1, 2, \ldots, n. \]  
(13)
From (5), there is no loss of generality in assuming \( \phi_1 \land \phi_2 = 1 \). Then for every minor \( \xi_a \) of \( \Theta \) with order \( m \), it follows that the inner factor of \( \phi_1 \) is a divisor of \( \xi_a \). In fact, if \( \xi_a \equiv 0 \), then it is obvious. Thus assume \( \xi_a \not\equiv 0 \) and \( \xi_a = \det \Theta_a \), where

\[
\Theta_a = \begin{bmatrix}
\theta_{a(1)1} & \cdots & \theta_{a(1)m} \\
\vdots & \ddots & \vdots \\
\theta_{a(m)1} & \cdots & \theta_{a(m)m}
\end{bmatrix}
\]

for \( 1 \leq a(1) < \cdots < a(m) < n \).

Then \( n > m \) implies that there is an \( l \) such that \( 1 < l < n \) and \( l \neq a(k) \) for \( k = 1, 2, \ldots, m \). From (12) it follows that

\[
- \phi_1 \begin{bmatrix}
\gamma_{a(1)l} \\
\vdots \\
\gamma_{a(m)l}
\end{bmatrix} = \Theta_a
\]

Now, since \( \det \Theta_a = \xi_a \not\equiv 0 \), there exists an \( m \times m \) matrix \( \Theta'_a \) over \( H^\infty \) such that \( \Theta_a \Theta'_a = \Theta'_a \Theta_a = \det \Theta_a = \xi_a \). Thus it is clear that

\[
- \phi_1 \Theta'_a
\]

which implies that \( \phi_1 \) is a divisor of \( \xi_a \omega_{1l} \) for \( i = 1, 2, \ldots, m \). Hence, by (13), \( \phi_1 \) is a divisor of

\[
\xi_a (\theta_1\omega_{1l} + \theta_2\omega_{2l} + \cdots + \theta_m\omega_{ml}) = \xi_a (\phi_2 - \phi_1 \gamma_{1l}).
\]

From \( \phi_1 \land \phi_2 = 1 \), this implies that the inner factor of \( \phi_1 \) is a divisor of \( \xi_a \). Thus it is a divisor of \( \land \xi_a = d_m \); this, from \( \phi_1 \land d_m = 1 \), yields that the inner factor of \( \phi_1 = 1 \) i.e., \( \phi_1 \) is outer. Since \( \phi_1 \) is a divisor of \( \xi_a \phi_2 \), there exist \( \xi_a \) in \( H^\infty \) such that \( \phi_1 \xi_a = \xi_a \phi_2 \). Since \( \Theta \) is inner \( \Sigma_a|\xi_a|^2 = 1 \) a.e. on the unit circle [5]. Thus,

\[
|\phi_2| = |\phi_1| \left( \sum |\xi_a|^2 \right)^{1/2} \leq |\phi_1| \left( \sum \|\xi_a\|^\infty \right) \text{ a.e.}
\]

implies that there exists a \( \phi \) in \( H^\infty \) such that \( \phi_1 \phi = \phi_2 \).

Thus we have \( \phi_1 (\phi - \Gamma) = \Theta \Omega \), which implies that

\[
\phi_1 (S(\Theta)) \left( \phi(S(\Theta)) - T \right) = 0
\]

and hence \( T = \phi(S(\Theta)) \). Thus we complete the proof of Theorem.

REFERENCES


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