QUASI-SIMILARITY OF WEAK CONTRACTIONS

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Abstract. Let $T$ be a completely nonunitary (c.n.u.) weak contraction (in the sense of Sz.-Nagy and Foias). We show that $T$ is quasi-similar to the direct sum of its $C_0$ part and $C_{11}$ part. As a corollary, two c.n.u. weak contractions are quasi-similar to each other if and only if their $C_0$ parts and $C_{11}$ parts are quasi-similar to each other, respectively. We also completely determine when c.n.u. weak contractions and $C_0$ contractions are quasi-similar to normal operators.

Recall that a contraction $T$ on the Hilbert space $H$ is called a weak contraction if its spectrum $\sigma(T)$ does not fill the open unit disc $D$ and $1 - T^*T$ is of finite trace. Contained in this class are all contractions $T$ with finite defect index $d_T \equiv \dim \text{rank}(1 - T^*T)^{1/2}$ and with $\sigma(T) \not= \overline{D}$ (cf. [9, p. 323]).

Assume that $T$ is a weak contraction which is also completely nonunitary (c.n.u.), that is, $T$ has no nontrivial reducing subspace on which $T$ is a unitary operator. For such a contraction, Sz.-Nagy and Foiaş obtained a $C_0$-$C_{11}$ decomposition and then found a variety of invariant subspaces which furnish its spectral decomposition (cf. [9, Chapter VIII]). In this note we are going to supplement other interesting properties of such contractions. We show that a c.n.u. weak contraction is quasi-similar to the direct sum of its $C_0$ part and $C_{11}$ part. Although the proof is not difficult, some of its interesting applications justify the elaboration here. An immediate corollary is that two such contractions are quasi-similar to each other if and only if their $C_0$ parts are quasi-similar and their $C_{11}$ parts are quasi-similar to each other. This is, in turn, used to show that two quasi-similar weak contractions have equal spectra. Another interesting consequence is that a c.n.u. weak contraction is quasi-similar to a normal operator if and only if its $C_0$ part is. The latter can be shown to be equivalent to the condition that its minimal function is a Blaschke product with simple zeros, thus completely settling the question when a c.n.u. weak contraction is quasi-similar to a normal operator.

Before we start to prove our main theorem, we provide some background work for our notations and terminology. The main reference is [9].

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1 This research was done while the author was visiting Indiana University during the summer of 1976.
Let $T$ be an arbitrary contraction on $H$. Let $H_0 = \{h \in H : T^n h \to 0\}$, $H'_0 = \{h \in H : T^{*n} h \to 0\}$, $H'_1 = H \ominus H_0$ and $H_1 = H \ominus H'_0$. Note that $H_0$ and $H'_0$ are invariant for $T$ and $T^*$, respectively. Consider the triangulations of $T$ with respect to the orthogonal decompositions $H = H_0 \oplus H'_1$ and $H = H_1 \oplus H'_0$:

$$T = \begin{bmatrix} T_0 & X \\ 0 & T'_1 \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} T_1 & Y \\ 0 & T'_0 \end{bmatrix}.$$ 

The triangulations are of type

$$\begin{bmatrix} C_0 & * \\ 0 & C_1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} C_1 & * \\ 0 & C_0 \end{bmatrix},$$

respectively (cf. [9, p. 73]). Recall that a contraction $T$ is of class $C_0$ (resp. $C_0$) if $T^n h \to 0$ (resp. $T^* h \to 0$) as $n \to \infty$ for all $h$ and $T$ is of class $C_1$ (resp. $C_1$) if $T^n h \to 0$ (resp. $T^{*n} h \to 0$) as $n \to \infty$ for all $h \neq 0$. $T$ is of class $C_{\infty}$ if $T \in C_0 \cap C_0$ and of class $C_{11}$ if $T \in C_1 \cap C_1$. A c.n.u. contraction $T$ is said to be of class $C_0$ if there exists a nonzero function $u \in H^\infty$ such that $u(T) = 0$. In this case we can choose $u$ to be a minimal inner function in the sense that $u$ is an inner function such that $u(T) = 0$ and $u$ divides (in $H^\infty$) every other function $v \in H^\infty$ for which $v(T) = 0$. Such a function is called a minimal function for $T$ and is denoted by $m_T$. If $T$ is a c.n.u. weak contraction, then in the previous triangulations $T_0$ is of class $C_0$ and $T_1$ is of class $C_{11}$, called the $C_0$ part and the $C_{11}$ part of $T$ (cf. [9, p. 331]). Note that in this case we have $H_0 \cup H_1 = H$ and $H_0 \cap H_1 = \{0\}$ (cf. [9, p. 332]). For arbitrary operators $T_1$, $T_2$ on $H_1$, $H_2$, respectively, $T_1 < T_2$ denotes that $T_1$ is a quasi-affine transform of $T_2$, that is, there exists a linear one-to-one and continuous transformation $S$ from $H_1$ onto a dense linear manifold in $H_2$ (called quasi-affinity) such that $ST_1 = T_2S$. $T_1$ and $T_2$ are quasi-similar if $T_1 < T_2$ and $T_2 < T_1$.

Our main theorem is the following:

**Theorem 1.** Let $T$ be a c.n.u. weak contraction on $H$. Let $T_0$ and $T_1$ be the $C_0$ part and $C_{11}$ part of $T$. Then $T$ is quasi-similar to $T_0 \oplus T_1$.

**Proof.** Let $S : H_0 \oplus H_1 \to H$ be defined by $S(h_0 \oplus h_1) = h_0 + h_1$. Certainly $T$ is a continuous linear transformation. Since $H_0 \cup H_1 = H$ and $H_0 \cap H_1 = \{0\}$, it is easily seen that $S$ is a quasi-affinity such that $S(T_0 \oplus T_1) = TS$. Thus $T_0 \oplus T_1 < T$. Note that $T^*$ is also a c.n.u. weak contraction and $T_0^*$ and $T_1^*$ are the $C_0$ and $C_{11}$ parts of $T^*$ (cf. [9, p. 332]). As above, we have $T_0^* \oplus T_1^* < T^*$. Hence $T < T_0^* \oplus T_1^*$, and $T_0 \oplus T_1 < T < T_0^* \oplus T_1^*$. Let $V$ be the quasi-affinity from $H_0 \oplus H_1$ to $H_0^* \oplus H_1^*$ such that $V(T_0 \oplus T_1) = (T_0^* \oplus T_1^*)V$. Since $T_0$ and $T_1$ are of class $C_0$ and $C_{11}$, respectively, it is easily seen that $VH_0 \subseteq H_0^*$. Say,

$$V = \begin{bmatrix} V_0 & Z \\ 0 & V_1 \end{bmatrix}$$
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is the corresponding triangulation. An easy calculation shows that $ZT_1 = T_0^*Z$. Since $T_1$ is of class $C_{11}$ and $T_0$ is of class $C_{00}$, we must have $Z = 0$ (cf. [4, Lemma 4.4]). Thus $V_0$ and $V_1$ are quasi-affinities satisfying $V_0T_0 = T_0^*V_0$ and $V_1T_1 = T_1^*V_1$. Hence $T_0 < T_0'$ and $T_1 < T_1'$. It follows from the uniqueness of the Jordan model for $C_0$ contractions that $T_0$ and $T_0'$ are quasi-similar to each other (cf. [2]). To show that $T_1$ is quasi-similar to $T_1'$, note that $T_1$ and $T_1'$, being $C_{11}$ contractions, are quasi-similar to unitary operators, say $U_1$ and $U_1'$, respectively. We have $U_1 < U_1'$. By a theorem of Douglas [5], $U_1$ and $U_1'$ are unitarily equivalent. Hence $T_1$ is quasi-similar to $T_1'$, and $T$ is quasi-similar to $T_0 \oplus T_1$.

An immediate corollary of Theorem 1 is

**Corollary 1.** Let $T_1$ and $T_2$ be c.n.u. weak contractions. Then $T_1$ and $T_2$ are quasi-similar to each other if and only if their $C_0$ parts are quasi-similar and their $C_{11}$ parts are quasi-similar to each other.

**Proof.** The sufficiency follows immediately from Theorem 1. The necessity can be proved by a similar argument as in Theorem 1.

In particular, for c.n.u. contractions with scalar-valued characteristic functions, we have

**Corollary 2.** For $j = 1, 2$, let $T_j$ be a c.n.u. contraction with the scalar-valued characteristic function $\psi_j \not\equiv 0$. Let $\psi_j = \psi_{ji}\psi_{je}$ be the canonical factorization into the product of its inner part $\psi_{ji}$ and outer part $\psi_{je}$, and let $E_j = \{e^{it}: |\psi_j(e^{it})| < 1\}$. Let

$$T_j = \begin{pmatrix} T_{j1} & X_j \\ 0 & T_{j2} \end{pmatrix}$$

be the triangulation of type

$$\begin{bmatrix} C_1 & * \\ 0 & C_0 \end{bmatrix}, \quad j = 1, 2.$$ 

Then the following are equivalent:

(i) $T_1$ is quasi-similar to $T_2$;

(ii) $T_{12}$ is quasi-similar to $T_{21}$ and $T_{12}$ is unitarily equivalent to $T_{22}$;

(iii) $\psi_{1i} = \psi_{2i}$ and $E_1$ and $E_2$ differ by a set of zero Lebesgue measure.

**Proof.** Since $T_1$ and $T_2$ are c.n.u. weak contractions, the equivalence of (i) and (ii) follows from Corollary 1. Note that $T_{ji}$ is quasi-similar to the multiplication by $e^{it}$ on the space $L^2(E_j)$ and $T_{j2}$ is unitarily equivalent to the compression of the shift $S(\psi_{ji})$ on $H^2 \ominus \psi_{ji}H^2$, $j = 1, 2$. Thus the equivalence of (ii) and (iii) follows immediately.

The equivalence of (i) and (iii) in Corollary 2 is compatible with the result of Kriete [7] that $T_1$ is similar to $T_2$ if and only if $\psi_1/\psi_2, \psi_2/\psi_1 \in H^\infty$ and $E_1$ and $E_2$ differ by a set of zero Lebesgue measure.

**Corollary 3.** Let $T_1$ and $T_2$ be c.n.u. weak contractions. If $T_1$ and $T_2$ are
quasi-similar to each other, then \( \sigma(T_1) = \sigma(T_2) \).

**Proof.** For \( j = 1, 2 \), let \( T_{j0} \) and \( T_{j1} \) be the \( C_0 \) part and \( C_{11} \) part of \( T_j \). By Corollary 1, \( T_{10} \) and \( T_{11} \) are quasi-similar to \( T_{20} \) and \( T_{21} \), respectively. Since the spectrum of a \( C_0 \) contraction is completely determined by its minimal function [9, p. 126], and \( T_{10} \) and \( T_{20} \) have the same minimal function, we have \( \sigma(T_{10}) = \sigma(T_{20}) \).

To show that \( \sigma(T_{11}) = \sigma(T_{21}) \), let \( U_j \) be the residual part of the minimal unitary dilation of \( T_{j1}, j = 1, 2 \) (cf. [9, p. 61]). Note that \( T_{j1} \) is quasi-similar to \( U_j \) and \( \sigma(T_{j1}) \) lies entirely on the unit circle (cf. [9, pp. 75, 328]). It follows that \( \sigma(T_{j1}) = \sigma(U_j) \) (cf. [9, pp. 311–312]). By Douglas’ theorem [5], \( U_1 \) and \( U_2 \) are quasi-similar implies they are unitarily equivalent. Thus \( \sigma(T_{11}) = \sigma(U_1) = \sigma(U_2) = \sigma(T_{21}) \). Since \( \sigma(T_j) = \sigma(T_{j0}) \cup \sigma(T_{j1}) \) [9, p. 332], we have \( \sigma(T_1) = \sigma(T_2) \), completing the proof.

We remark that the proof can be modified to show that quasi-similar weak contractions (not necessarily c.n.u.) have equal spectra. This result is not new. It also follows from the facts that weak contractions are decomposable [6] and quasi-similar decomposable operators have equal spectra [3]. However, our proof seems more direct.

In the remaining part of this note we are concerned with the question when a c.n.u. weak contraction is quasi-similar to a normal operator. The next theorem reduces the problem to the \( C_0 \) part of the c.n.u. weak contraction.

**Theorem 2.** Let \( T \) be a c.n.u. weak contraction on \( H \). Let

\[
T = \begin{bmatrix}
T_0 & X \\
0 & T_1'
\end{bmatrix}
\]

be the triangulation of type

\[
\begin{bmatrix}
C_0 & * \\
0 & C_1
\end{bmatrix}
\]

on the (orthogonal) decomposition \( H = H_0 \oplus H_1' \). Then \( T \) is quasi-similar to a normal operator if and only if \( T_0 \) is.

**Proof.** The sufficiency follows trivially from Theorem 1. To prove the necessity, we may assume that \( T \) is quasi-similar to a normal operator \( N \) on the space \( K \) with \( \|N\| \leq \|T\| < 1 \) (cf. [1, Proof of the sufficiency part of Theorem]). Let \( K = K_1 \oplus K_2 \) be the direct sum of reducing subspaces for \( N \) such that \( N_1 = N|K_1 \) is c.n.u. and \( N_2 = N|K_2 \) is unitary. Let \( S \) be the quasi-affinity from \( H \) to \( K \) such that \( ST = NS \). Since \( T_0 \) is of class \( C_0 \) and \( N_2 \) is of class \( C_1 \), it is easily seen that \( SH_0 \subseteq K_1 \). Note that \( \overline{SH_0} \) is an invariant subspace for \( N_1 \). Let \( N_1' = N_1|\overline{SH_0} \). Then \( S_1 = S|H_0 \) is a quasi-affinity from \( H_0 \) to \( \overline{SH_0} \) satisfying \( S_1T_0 = N_1'S_1 \). Since \( T_0 \) is of class \( C_0 \), so is \( N_1' \) (cf. [9, p. 125]). By the uniqueness of the Jordan model for \( C_0 \) contractions, we have \( T_0 \) is quasi-similar to \( N_1' \) (cf. [2]). Since \( N_1' \) is subnormal and \( \sigma(N_1') \) has planar
area zero (cf. [9, p. 126]), it follows from Putnam’s theorem [8] that \( N' \) is normal. This completes the proof.

Notice that Theorem 2 is compatible with the result that \( T \) is similar to a normal operator if and only if \( T_0 \) is similar to a normal operator and \( T' \) is similar to a unitary operator. This is true even for an arbitrary c.n.u. contraction (cf. [10, Theorem 3]).

Since the \( C_0 \) part of a c.n.u. weak contraction is a \( C_0 \) contraction, the next theorem furnishes the complete solution to the previously posed question.

**Theorem 3.** Let \( T \) be a \( C_0 \) contraction on the space \( H \) with the minimal function \( m_T \). Then \( T \) is quasi-similar to a normal operator if and only if \( m_T \) is a Blaschke product with simple zeros.

**Proof.** Necessity. Let \( T \) be quasi-similar to the normal operator \( N \) on the space \( K \) and let \( S \) be the quasi-affinity from \( H \) to \( K \) such that \( ST = NS \). As before we may assume that \( \|N\| < \|T\| < 1 \) (cf. [1]). Now we show that \( N \) must be c.n.u. Indeed, for any \( k \in K \) and \( \epsilon > 0 \), let \( h \in H \) be such that \( \|k - Sh\| < \epsilon \). Since \( ST^n h = N^n Sh \to 0 \) as \( n \to \infty \), we have \( \|N^n Sh\| < \epsilon \) for all \( n > N_0 \). Hence

\[
\|N^n k\| < \|N^n k - N^n Sh\| + \|N^n Sh\| < \|N\| \|k - Sh\| + \|N^n Sh\| < \epsilon + \epsilon = 2\epsilon \quad \text{for all } n > N_0.
\]

This shows that \( N^n k \to 0 \) for all \( k \in K \) and hence \( N \) is c.n.u. Since \( N \) is quasi-similar to a \( C_0 \) contraction, \( N \) is also a \( C_0 \) contraction with the same minimal function \( m_N = m_T \) (cf. [9, p. 125]). Let \( m_T = Bs \), where

\[
B(\lambda) = \prod_i \frac{\bar{\lambda}_i}{|\lambda_i|} \left( \frac{-\lambda_i}{1 - \bar{\lambda}_i \lambda} \right)^{n_i}
\]

is a Blaschke product and \( s \) is a singular function. Note that \( \lambda_i \) is a characteristic value of \( N \) with index \( n_i \) (cf. [9, p. 135]). Since \( N \) is a normal operator, \( n_i = 1 \) for all \( i \). Let \( K_i \) be the corresponding eigenspace. Then \( \bigvee_i K_i \) reduces \( N \) and the normal operator \( N_1 = N|(\bigvee_i K_i)^\perp \) has no eigenvalue. Hence the minimal function of the \( C_0 \) contraction \( N_1 \) must be \( s \) (cf. [9, p. 129]). It follows that \( s(N_1) \) is contained in the unit circle, and thus \( N_1 \) is a unitary operator. Since \( N \) is c.n.u., we must have \( (\bigvee_i K_i)^\perp = \{0\} \) and \( K = \bigvee_i K_i \). Hence \( m_T = B \) is a Blaschke product with simple zeros (cf. [9, p. 135]).

Sufficiency. Assume that \( m_T \) is a Blaschke product with simple zeros, say,

\[
m_T(\lambda) = \prod_i \frac{\bar{\lambda}_i}{|\lambda_i|} \frac{-\lambda_i}{1 - \bar{\lambda}_i \lambda},
\]

where the distinct \( \lambda_i \)'s satisfy \( |\lambda_i| < 1 \) and \( \sum_i(1 - |\lambda_i|) < \infty \). For each \( i \) let \( H_i = \{ h \in H : (T - \lambda_i) h = 0 \} \). Then \( T|H_i \) is a normal operator and the system \( \{ H_i \}_{i=1}^{\infty} \) of invariant subspaces satisfies

\[
H = H_i + \bigvee_{j \neq i} H_j \quad \text{for each } i, \quad \text{and} \quad \bigcap_i \left( \bigvee_{j > i} H_j \right) = \{0\}.
\]
That is, $\{H_i\}_{i=1}^{\infty}$ is a basic system of invariant subspaces for $T$. By a result of Apostol [1], $T$ is quasi-similar to a normal operator, completing the proof.

REFERENCES


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