A THEOREM ON $C^*$-EMBEDDING

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ABSTRACT. Theorem. In a totally nonmeager and regular space, every countable intersection of open, normal, $C^*$-embedded subsets is normal and $C^*$-embedded.

A subspace $G$ of a topological space $S$ is called $C^*$-embedded if every bounded continuous $f: G \to \mathbb{R}$ extends continuously to $S$. This note gives a short and elementary proof of the theorem in the abstract. This extends a result of E. Aron ([1], otherwise unpublished) which asserts that in a compact space, every countable intersection of dense, $C^*$-embedded, open $F_\sigma$ subsets is $C^*$-embedded. The original proof in [1] is rather long and nonelementary. The interest of this theorem and its relevance to certain problems in rings of continuous functions are discussed in [4, §6], [5], and [6, §5.8].

A space is called totally nonmeager (see [2, p. 252]) if every closed subspace is second category (i.e., nonmeager) relative to itself.

Proof of the Theorem. Suppose $S$ is a totally nonmeager, regular space and $G_1, G_2, \ldots$ are open, normal, $C^*$-embedded subsets. It suffices to prove disjoint closures in $S$ (for then, since $G_1$ is normal and $\overline{Z_1} \cap \overline{Z_2} \cap G_1 = \emptyset$, $G$ would be normal and $C^*$-embedded in $G_1$ [3, Theorem 1.17]). Put

$$K = \overline{Z_1} \cap \overline{Z_2}.$$ 

Clearly $G \cap K = \emptyset$. Suppose (toward a contradiction) that $K \neq \emptyset$. Fix $n$, pick any closed neighborhood $F$ of a point in $K$, and set $A_i = G_n \cap \overline{Z_i} \cap F$, $i = 1, 2$. Then

$$\emptyset \neq \overline{Z_1} \cap \overline{F} \cap \overline{Z_2} \cap \overline{F} \subset A_1 \cap A_2,$$

so the relatively closed subsets $A_1, A_2$ of the normal, $C^*$-embedded set $G_n$ do not have disjoint closures. Thus

$$\emptyset \neq A_1 \cap A_2 = G_n \cap K \cap F.$$ 

Since $F$ was arbitrary and $K$ is regular, $G_n \cap K$ is a dense relatively open subset of the nonmeager space $K$ (for all $n$). Thus $\emptyset \neq \cap_{n=1}^{\infty}(G_n \cap K) = G \cap K$, a contradiction.

REFERENCES


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