LOCAL COMPACTNESS AND
HEWITT REALCOMPACTIFICATIONS OF PRODUCTS

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Abstract. In this note we prove McArthur's conjecture [6]: If card X is
nonmeasurable and if \( v(X \times Y) = vX \times vY \) holds for each space Y, then X
is locally compact. Consequently, we can completely characterize the class
of all spaces X such that for each space Y, \( v(X \times Y) = vX \times vY \) holds.

1. Introduction. All spaces considered in this note will be completely regular
Hausdorff. For a space X, vX denotes the Hewitt realcompactification of X,
and the symbolism \( v(X \times Y) = vX \times vY \) means that \( X \times Y \) is C-embedded
in \( vX \times vY \). Following [6], let \( R \) denote the class of all spaces X such that for
each space Y, \( v(X \times Y) = vX \times vY \) holds. It is known that a locally compact
realcompact space of nonmeasurable cardinal is a member of \( R \) and that
every member of \( R \) is realcompact (Comfort [1, Corollary 2.2], McArthur [6,
Theorem 5.2]). In [6], McArthur conjectured that if card X is nonmeasurable
and X is a member of \( R \), then X is locally compact. The main purpose of this
note is to establish his conjecture positively. More precisely, we can prove the
following theorems. The implication (a) \( \rightarrow \) (b) of Theorem 1 was proved by
Comfort [1].

Theorem 1. For a space X of nonmeasurable cardinal the following conditions
are equivalent:
(a) X is locally compact.
(b) \( X \times Y \) is C-embedded in \( X \times vY \) for each space Y.

Theorem 2. For a space X of nonmeasurable cardinal the following conditions
are equivalent:
(a) X is locally pseudocompact.
(b) \( X \times Y \) is C-embedded in \( X \times vY \) for each k-space Y.

We remark that, in Theorems 1 and 2, the assumption "card X is non-
measurable" is useful only for the implication (a) \( \rightarrow \) (b). Combining these
theorems with the results of Comfort and McArthur, quoted above, and
Hušek [4, Theorem 3], we have the following theorems.

Theorem 3. For a space X the following conditions are equivalent:
(a) $X$ is locally compact, realcompact and card $X$ is nonmeasurable.

(b) $u(X \times Y) = vX \times vY$ holds for each space $Y$.

**THEOREM 4.** For a space $X$ the following conditions are equivalent:

(a) $vX$ is locally compact and card $X$ is nonmeasurable.

(b) $u(X \times Y) = vX \times vY$ holds for each $k$-space $Y$.

For the notions of locally pseudocompact spaces and $k$-spaces see [1]. For an ordinal $\alpha$, we denote by $W(\alpha)$ the set of all ordinals less than $\alpha$ topologized with order topology, and by $\omega_0$ the first infinite ordinal. Other terms can be found in [3].

2. **Proofs of theorems.**

**Proof of Theorem 1.** (a) $\rightarrow$ (b). This is the result of Comfort [1, Theorem 2.1]. (b) $\rightarrow$ (a). Suppose, on the contrary, that $X$ is not locally compact at $x_0 \in X$. Let $\{G_\lambda \mid \lambda \in \Lambda\}$ be a neighborhood base at $x_0$ in $X$. Then, for each $\lambda \in \Lambda$, $cl_x G_\lambda$ is not compact, and thus there exists a point $x_\lambda \in cl_{\beta X} G_\lambda \cap (\beta X - X)$, where $\beta X$ is the Stone-Čech compactification of $X$. For each $\lambda \in \Lambda$, let $\{G(\lambda, \sigma) \mid \sigma \in \Sigma_\lambda\}$ be a neighborhood base at $x_\lambda$ in $\beta X$. For each $\sigma \in \Sigma_\lambda$, we can choose a point $x(\lambda, \sigma) \in X$ and an open set $H(\lambda, \sigma)$ in $X$ such that $x(\lambda, \sigma) \in H(\lambda, \sigma) \subset G(\lambda, \sigma) \cap G_\lambda$. Let $s_\lambda$ be an ideal point, and set $S_\lambda = \Sigma_\lambda \cup \{s_\lambda\}$, topologized as follows: Each point of $\Sigma_\lambda$ is isolated and $\{J(\lambda, \sigma) \mid \sigma \in \Sigma_\lambda\}$ is a neighborhood base at $s_\lambda$, where $J(\lambda, \sigma) = \{s_\lambda\} \cup \{\tau \in \Sigma_\lambda \mid G(\lambda, \sigma) \supset G(\lambda, \tau)\}$. Let $n$ be a regular cardinal greater than sup(card $\Sigma_\lambda \mid \lambda \in \Lambda$), and let $\omega_\alpha$ be the initial ordinal of $n$. For each $\lambda \in \Lambda$, let

$$T_{\lambda(1)} = \{((\lambda(1), \gamma, \beta)) \mid \gamma < \omega_\alpha, \beta < \omega_0\}$$

be the copy of $W((\omega_\alpha + 1) \times (\omega_0 + 1))$, and let

$$T_{\lambda(2)} = \{((\lambda(2), \gamma, s)) \mid \gamma < \omega_\alpha, s \in S_\lambda\}$$

be the copy of $W((\omega_\alpha + 1) \times S_\lambda)$. By identifying a point $(\lambda(1), \gamma, \omega_\alpha)$ with $(\lambda(2), \gamma, s_\lambda)$ for $\gamma < \omega_\alpha$, we have a quotient space $T_\lambda$ and a quotient map $f_\lambda : T_{\lambda(1)} \oplus T_{\lambda(2)} \rightarrow T_\lambda$, where $A \oplus B$ denotes the topological sum of $A$ and $B$. Let us set $Z = \bigoplus \{T_\lambda \mid \lambda \in \Lambda\}$, and let $Y_0$ be the quotient space obtained from $Z$ by collapsing a set $\{f_\lambda((\lambda(1), \omega_\alpha, \beta)) \mid \lambda \in \Lambda\}$ to a single point $y(\beta) \in Y_0$ for $\beta < \omega_0$. Let $g : Z \rightarrow Y_0$ be the quotient map, and set $h_\lambda = g \circ f_\lambda$ for each $\lambda \in \Lambda$. Then $y(\omega_0) = h_\lambda((\lambda(2), \omega_\alpha, s_\lambda))$ for each $\lambda \in \Lambda$. Let us set $Y = Y_0 - \{y_0\}$, where $y_0 = y(\omega_0)$. We shall now prove that $Y \subset vY$ by showing that $Y$ is $C$-embedded in $Y_0$. Let $\phi$ be a real-valued continuous function on $Y$. For each $\lambda \in \Lambda$, by the same argument as in [3, 8.20], there is $\gamma_\lambda \in W(\omega_\alpha)$ such that $\theta_\lambda = \phi \circ (h_\lambda h_\lambda^{-1}(Y))$ takes on the constant value $t_\lambda$ on $\{(\lambda(1), \gamma, \omega_0) \mid \gamma_\lambda < \gamma < \omega_\alpha\} \cup \{(\lambda(2), \gamma, s_\lambda) \mid \gamma_\lambda < \gamma < \omega_\alpha\}$. Since

$$\theta_\lambda((\lambda(1), \omega_\alpha, \beta)) = \theta_\mu((\mu(1), \omega_\alpha, \beta))$$

for $\lambda, \mu \in \Lambda$ and for each $\beta < \omega_0$, we have $t_\lambda = t_\mu$ for $\lambda, \mu \in \Lambda$. Extend $\phi$ over $Y_0$ by setting $\phi(y_0) = t_\lambda$. Then it is easy to see that the extension $\phi$ is
continuous. Thus $Y$ is $C$-embedded in $Y_0$ and hence $Y_0 \subseteq vY$. It remains to show that $X \times Y$ is not $C$-embedded in $X \times vY$. For each $\lambda \in \Lambda$ and each $\sigma \in \Sigma_\lambda$, let us set

$$y(\lambda, \sigma) = h_\lambda((\lambda(2), \omega_\sigma, \sigma)),$$

$$K(\lambda, \sigma) = h_\lambda\left(\left\{ (\lambda(2), \gamma, \sigma) | \gamma \leq \omega_\sigma \right\}\}.\right.$$

And let us set

$$p(\lambda, \sigma) = (x(\lambda, \sigma), y(\lambda, \sigma)) \in X \times Y,$$

$$L(\lambda, \sigma) = H(\lambda, \sigma) \times K(\lambda, \sigma) \subset X \times Y,$$

$$E = \{ L(\lambda, \sigma) | \lambda \in \Lambda, \sigma \in \Sigma_\lambda \}.$$

Then $L(\lambda, \sigma)$ is a neighborhood at $p(\lambda, \sigma)$ in $X \times Y$. Now we show that $E$ is discrete in $X \times Y$. To do this, let $p = (x, y) \in X \times Y$; then $y = h_\lambda((\mu(i), \delta, t))$ for some $\mu \in \Lambda$, $i \in \{1, 2\}$, $\delta \leq \omega_\alpha$ and $t \in W(\omega_\alpha + 1) \oplus S_\mu$. If $t \in W(\omega_\alpha + 1)$ and $t < \omega_\alpha$, then

$$V(y) = \bigcup \{ h_\lambda(T_{\lambda(i)} | \lambda \in \Lambda) \} \cap Y$$

is a neighborhood at $y$ in $Y$ such that $V(y) \cap K(\lambda, \sigma) = \emptyset$ for each $\lambda \in \Lambda$ and each $\sigma \in \Sigma_\lambda$, and hence $X \times V(y)$ is a neighborhood at $p$ which meets no member of $E$. If $t = \omega_\alpha$ or $s_\mu$, then there exist $\tau \in \Sigma_\mu$ and a neighborhood $V(x)$ at $x$ such that $V(x) \cap G(\mu, \tau) = \emptyset$. If we set

$$V(y) = \bigcup \{ (\mu(1), \gamma, \beta) | \gamma \leq \delta, \beta \leq \omega_0 \}$$

$$\cup \{ (\mu(2), \gamma, s) | \gamma \leq \delta, s \in J(\mu, \tau) \};$$

then $V(y)$ is a neighborhood at $y$ in $Y$ such that $V(x) \times V(y)$ meets no member of $E$. If $t \in \Sigma_\mu$, then $X \times K(\mu, t)$ is a neighborhood at $p$ which meets only $L(\mu, t)$. Hence $E$ is discrete in $X \times Y$. For each $\lambda \in \Lambda$ and each $\sigma \in \Sigma_\lambda$, there is a real-valued continuous function $\psi(\lambda, \sigma)$ on $X \times Y$ such that $\psi(\lambda, \sigma)(p(\lambda, \sigma)) = 0$ and $\psi(\lambda, \sigma)(q) = 1$ for each $q \in (X \times Y) - L(\lambda, \sigma)$. If we define a function $\psi$ by

$$\psi(q) = \inf \{ \psi(\lambda, \sigma)(q) | \lambda \in \Lambda, \sigma \in \Sigma_\lambda \}, \quad q \in X \times Y,$$

then $\psi$ is continuous, since $E$ is discrete. For our purpose, it suffices to show that $\psi$ admits no continuous extension to the point $p_0 = (x_0, y_0) \in X \times vY$. Let $U$ be a given neighborhood at $p_0$. There exist $\mu \in \Lambda$ and a neighborhood $V(y_0)$ at $y_0$ in $Y_0$ such that $p_0 \in G_\mu \times V(y_0) \subset U$. Then $y(\mu, \tau) \in V(y_0)$ for some $\tau \in \Sigma_\mu$, and hence $p(\mu, \tau) \in U$ and $\psi(p(\mu, \tau)) = 0$. On the other hand, $y(\beta)$ is in $V(y_0)$ for some $\beta < \omega_0$, and then $q = (x_0, y(\beta)) \in U$ and $\psi(q) = 1$. This shows that $\psi$ does not extend continuously to $p_0$. Hence the proof is completed.

Before proving Theorem 2, we prove the implication (a) $\rightarrow$ (b) of Theorem 4, which slightly improves a theorem of Comfort [1, Theorem 2.4]. We denote
by \( \mu X \) the topological completion of \( X \) (i.e., the completion of \( X \) with respect to its finest uniformity).

**Proof of Theorem 4.** (a) \( \rightarrow \) (b). Assume that \( \nu X \) is locally compact and \( \text{card} \ X \) is nonmeasurable. Let \( Y \) be a \( k \)-space. Then, by [1, Theorem 2.1], \( \nu X \times Y \) is \( C \)-embedded in \( \nu X \times \nu Y \). Since \( \nu X \) is locally compact, by [5, Theorem 1.5], we have \( \nu X = \mu X \). Hence \( \mu(X \times Y) = \mu X \times \mu Y \) holds by [5, Theorem 2.3], and so \( X \times Y \) is \( C \)-embedded in \( \mu X \times Y \left( = \nu X \times Y \right) \). Thus we have \( \nu(X \times Y) = \nu X \times \nu Y \).

**Proof of Theorem 2.** (a) \( \rightarrow \) (b). Let \( X \) be a locally pseudocompact space of nonmeasurable cardinal and let \( Y \) be a \( k \)-space. Now it suffices to show that for each pseudocompact subset \( S \) of \( X \), \( S \times Y \) is \( C \)-embedded in \( S \times \nu Y \). To see this, let \( S \) be a given pseudocompact subset of \( X \), then we have \( \nu S = \beta S \) by [3, 8A4]. Thus \( \nu(S \times Y) = \nu S \times \nu Y \) holds by Theorem 4, (a) \( \rightarrow \) (b) proved above, and hence \( S \times Y \) is \( C \)-embedded in \( S \times \nu Y \). (b) \( \rightarrow \) (a). Suppose on the contrary that \( X \) is not locally pseudocompact at \( x_0 \in X \). Let \( \{ G_\lambda | \lambda \in \Lambda \} \) be a neighborhood base at \( x_0 \). Then, for each \( \lambda \in \Lambda \), \( \text{cl}_X G_\lambda \) is not pseudocompact, and thus we can find a countable decreasing family \( \{ G(\lambda, \sigma) | \sigma \in \Sigma_\lambda \} \) of open sets in \( X \) such that \( \bigcap \{ \text{cl}_X G(\lambda, \sigma) | \sigma \in \Sigma_\lambda \} = \emptyset \) and \( G(\lambda, \sigma) \subseteq G_\lambda \) for each \( \sigma \in \Sigma_\lambda \). Let us set \( H(\lambda, \sigma) = G(\lambda, \sigma) \), and choose a point \( x(\lambda, \sigma) \in H(\lambda, \sigma) \). We construct \( Y_0 \) and \( Y \) quite similarly to the proof of Theorem 1. Examining the process, one sees that then each \( S_\lambda \) is compact, and hence \( Z \) is locally compact. Since every quotient space and open subspace of a \( k \)-space is a \( k \)-space, \( Y \) is a \( k \)-space. Therefore, by pursuing the proof of Theorem 1, we have Theorem 2.

To prove the implication (b) \( \rightarrow \) (a) of Theorems 3 and 4, we need a theorem of Hušek [4, Theorem 3]. His theorem can be restated as follows:

**Hušek’s Theorem.** For a space \( X \) the following conditions are equivalent:

(a) \( \text{card} \ X \) is nonmeasurable.

(b) \( \nu(X \times Y) = \nu X \times \nu Y \) holds for each discrete space \( Y \).

**Proof of Theorem 3.** (a) \( \rightarrow \) (b) is the result of Comfort quoted in the introduction. (b) \( \rightarrow \) (a). By Hušek’s theorem, \( \text{card} \ X \) is nonmeasurable. It follows from Theorem 1 and [6, Theorem 5.2] that \( X \) is locally compact and realcompact.

**Proof of Theorem 4.** (b) \( \rightarrow \) (a). Since a discrete space is a \( k \)-space, by Hušek’s Theorem, \( \text{card} \ X \) is nonmeasurable. By Theorem 2, \( \nu X \) is locally pseudocompact, and hence is locally compact, because every pseudocompact realcompact space is compact (cf. [3, 8E1]).

3. **Remarks.** (1) If \( \nu X \) is locally compact, then \( X \) is locally pseudocompact, but the converse is false (see [1]).

(2) The space \( Y \) constructed in the proof of Theorems 1 and 2 and [6, Theorem 5.2] is 0-dimensional (i.e., \( \text{ind} \ Y = 0 \)). Hence all theorems in this note remain true if “for each \( (k-) \) space \( Y \)” is replaced by “for each 0-dimensional \( (k-) \) space \( Y \)”.
(3) A space similar to the space $S_x$ in the proof of Theorem 1 was used in [6] to show that every member of $\mathcal{R}_x$ is realcompact.

(4) A space $X$ is said to be topologically complete if it is complete with respect to its finest uniformity (i.e., $X = \mu X$). In [7], Morita proved that if $X$ is locally compact topologically complete, then $\mu(X \times Y) = \mu X \times \mu Y$ holds for each space $Y$, and Isiwata [5] proved that if $\mu(X \times Y) = \mu X \times \mu Y$ holds for each space $Y$, then $X$ is topologically complete (cf. also [8]). Hence the analogous results of Theorems 1-4 remain true, with no cardinality conditions, for topological completions (in this case, we need to use [5, Theorem 2.3], [7, Theorem 3.1] and [2, Lemma 3.1] instead of Theorem 4, (a) $\rightarrow$ (b), [3, 8A4] and [3, 8E1], respectively).

**Added in Proof.** Recently, Blair and Hager (z-embedding in $\beta X \times \beta Y$, *Set theoretic topology*, Academic Press, New York, 1977) asked whether the following condition (d') implies that $X \times Y$ is z-embedded in $\beta X \times \beta Y$ (i.e., each zero-set of $X \times Y$ is the trace on $X \times Y$ of a zero-set of $\beta X \times \beta Y$):

(d') For every real-valued continuous function $f$ on $X \times Y$ and every $\varepsilon > 0$, there is a countable open rectangular cover $\{G_n\}$ of $X \times Y$ such that $\sup \{|f(p) - f(q)| \mid p, q \in G_n\} < \varepsilon$ for each $n$.

In the same paper, they proved that if $X$ has a countable base, then $X \times Y$ satisfies (d') for each space $Y$, and that if $X \times Y$ is z-embedded in $\beta X \times \beta Y$, then $\nu(X \times Y) = \nu X \times \nu Y$ holds. From these facts, since there exists a nonlocally compact space with a countable base, Theorem 3 answers this question negatively. Furthermore, combining Theorem 3 with their results (3.2, 3.3), we obtain: $X$ is a locally compact space with a countable base if and only if $X \times Y$ is z-embedded in $\beta X \times \beta Y$ for each space $Y$.

**REFERENCES**


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