AN IMPROVEMENT THEOREM FOR DESCARTES SYSTEMS

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Abstract. An improvement (or comparison) theorem is proved for certain linear combinations of functions from a Descartes system. This theorem can then be applied to prove a conjecture of Lorentz, as well as more general results.

1. Introduction. The results in this paper were motivated by a problem posed by G. G. Lorentz. Lorentz was interested in minimizing \( \| x^N - \sum_{i=1}^{k} a_i x^{\lambda_i} \| \) where the \( \lambda_i \) are integers, \( 0 < \lambda_i < N \), the \( a_i \) are real numbers, \( k < N \), and \( \| \cdot \| \) is the supremum norm on \([0, 1]\). It was conjectured by Lorentz that for given integers \( k \) and \( N \) the set of exponents \( \lambda_i \) which produced the smallest error is \( \lambda_i = N - i, i = 1, \ldots, k \). This was proved in [1] by noting that the kernel \( K(x, y) = x^y \) is extended totally positive (ETP) on \((0, \infty) \times (-\infty, \infty)\).

Subsequently, A. Pinkus [5] observed that this result is valid when \( \| \cdot \| \) is any \( L_p \) norm, \( 1 < p < \infty \). His proof relied on the fact that \( \{ x^i \}_{i=0}^{N} \) is a Descartes system on \((0, 1)\).

These results are very striking since one might not expect the same set of exponents to produce the smallest error in all the \( L_p \) norms. The purpose of this paper is to expose the basic property of Descartes systems from which these and more general results flow, namely

**Theorem 1.** Let \( \{ u_i \}_{i=1}^{N} \subset C(c, d) \) be a Descartes system on \((c, d)\). Let an integer \( k < N - 2 \) be given along with integers \( N > \lambda_i > \gamma_i > 1 \) for \( i = 1, \ldots, k \). Suppose that \( c < x_1 < \cdots < x_k < d \),

\[
p = u_N + \sum_{i=1}^{k} a_i u_{\lambda_i}, \quad q = u_N + \sum_{i=1}^{k} b_i u_{\gamma_i},
\]

and

\[
0 = p(x_i) = q(x_i), \quad i = 1, \ldots, k.
\]

Then \( |p(x)| < |q(x)| \) for all \( x \in (c, d) \), with strict inequality if \( x \neq x_i, \ i = 1, \ldots, k \), provided that \( p \neq q \).

This theorem is reminiscent of the “improvement” theorems of Karlin [3].
§2 contains the relevant definitions, a preliminary lemma, and the proof of Theorem 1. In addition, we state an extension of Theorem 1. §3 relates Theorem 1 and its extension to certain approximation results.

2. Proof of Theorem 1 and extensions. We begin with some necessary definitions and notations. A set of functions \( \{u_i\}_{i=1}^{N} \subset C(c, d) \) will be called a Descartes system on \((c, d)\) [2, pp. 25–27] provided there is an \( e_k = \pm 1 \) so that

\[
0 < e_k \det \begin{bmatrix}
u_{\lambda_1}(t_1) & u_{\lambda_2}(t_1) & \cdots & u_{\lambda_k}(t_1) \\
\vdots & \vdots & & \vdots \\
u_{\lambda_1}(t_k) & u_{\lambda_2}(t_k) & \cdots & u_{\lambda_k}(t_k)
\end{bmatrix}
\]

whenever \( 1 < \lambda_1 < \cdots < \lambda_k < N, \ c < t_1 < \cdots < t_k < d, \) and \( 1 < k < N. \)

The next lemma can be easily verified via Cramer’s rule.

**Lemma.** Let \( \{u_i\}_{i=1}^{N} \) be a Descartes system on \((c, d)\), \( 2 < m < N, \) integers \( 1 < \lambda_1 < \cdots < \lambda_m < N, \) and \( c < x_1 < \cdots < x_m-1 < d \) be given. Suppose that \( p = 2^{m-1} \sum a_i u_i \) is not zero but \( p(x_i) = 0 \) for \( i = 1, \ldots, m-1. \) Then

(i) \( p(x) = 0 \) only if \( x = x_i, \ 1 < i < m-1. \)

(ii) \( p \) changes sign at each \( x_i. \)

(iii) \( a_i a_{i+1} < 0 \) for \( i = 1, \ldots, m-1. \)

(iv) \( a_m p(x) e_m e_{m-1} > 0 \) for \( x_{m-1} < x < d. \)

This lemma is the key to proving Theorem 1. It provides the necessary information concerning the “orientation” of elements in the span of a Descartes system which vanish maximally.

We now proceed to prove Theorem 1. It is clear that we need only consider \( p \) and \( q \) of the form

\[
p = u_N + \sum_{i=1}^{k} a_i u_{\lambda_i} + a u_{\lambda}, \quad q = u_N + \sum_{i=1}^{k} b_i u_{\lambda_i} + b u_{\gamma}
\]

where \( 1 < \lambda_1 < \cdots < \lambda_{j-1} < \gamma < \lambda < \lambda_{j+1} < \cdots < \lambda_k < N, \) since the general result may be inferred from this case by making a finite number of pairwise comparisons.

With \( p \) and \( q \) as above the proof proceeds by showing that \( p \) and \( q \) have the same “orientation” (i.e. sign structure) but that \( p - q \) has opposite “orientation”. This will complete the proof. By hypothesis \( p \) and \( q \) have zeros at \( x_1, \ldots, x_k \) and hence part (iv) of the Lemma implies that

\[
p(x) e_{k+1} e_k > 0 \quad \text{and} \quad q(x) e_{k+1} e_k > 0
\]

for \( x_k < x < d. \) Furthermore,

\[
p - q = \sum_{i=1}^{k} c_i u_{\lambda_i} + a u_{\lambda} - b u_{\gamma}
\]
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has \( k + 1 \) terms and also vanishes maximally at \( x_1, \ldots, x_k \). Thus, the coefficient of the leading term for \( p - q \) (i.e. \( c_k \) if \( j \neq k \)) is negative by part (iii) of the Lemma (since \( p \) and \( p - q \) have the same coefficient for \( u_k \)). By part (iv) of the Lemma we have

\[
(p - q)(x)\varepsilon_{k+1}\varepsilon_k < 0, \quad x_k < x < d. \tag{2.3}
\]

Combining (2.2) and (2.3) we have, for \( x_k < x < d \)

\[
0 < p(x) < q(x) \quad \text{if} \quad \varepsilon_{k+1}\varepsilon_k > 0
\]

or

\[
0 > p(x) > q(x) \quad \text{if} \quad \varepsilon_{k+1}\varepsilon_k < 0.
\]

In any case we see that \( |p(x)| < |q(x)| \) for \( x_k < x < d \).

Using part (ii) of the Lemma we note that \( p, q, \) and \( p - q \) all change sign at \( x_k \) and hence for \( x_{k-1} < x < x_k \) we have \( |p(x)| < |q(x)| \). Applying this argument repeatedly completes the proof of Theorem 1.

Theorem 1 may be generalized as follows:

**Theorem 2.** Let \( \{u_i\}_{i=1}^N \subset C(a, b) \) be a Descartes system on \((a, b)\). Let nonnegative integers \( k, l, m, \) and \( \alpha \) be given satisfying \( l + m = k, \ 1 < \alpha - l, \) and \( \alpha + m < N \). Suppose that \( a < x_1 < \cdots < x_k < b \)

\[
p = u_\alpha + \sum_{i=1}^k a_i u_{\lambda_i}, \quad q = u_\alpha + \sum_{i=1}^k b_i u_{\lambda_i},
\]

and

\[
0 = p(x_i) = q(x_i)
\]

where \( 1 < \gamma_i < \lambda_i < \alpha \) for \( i = 1, \ldots, l \) and \( \alpha < \lambda_i < \gamma_i < N \) for \( i = l + 1, \ldots, k \). Then \( |p(x)| < |q(x)| \) for all \( x \in (a, b) \) with strict inequality if \( x \neq x_i, \ i = 1, \ldots, k \) provided \( p \neq q \).

The proof of this theorem may be safely omitted since it is quite similar to the proof of Theorem 1.

3. Applications. Throughout this section we will assume that \( \{u_i\}_{i=1}^N \subset C(a, b) \) is a Descartes system on \((a, b)\) and that \( \{u_i\}_{i=1}^N \subset L_p(a, b), \ 1 < p < \infty \), whenever the \( L_p \) norm is discussed (when \( p = \infty \) we really mean \( C[a, b] \) with the supremum norm and the \( \{u_i\}_{i=1}^N \) from a Descartes system on the closed interval \([a, b]\)). We will denote by \( \|f\|_p \) the integral \( (\int_a^b |f(t)|^p \ dt)^{1/p} \) with appropriate modification if \( p = \infty \).

Let \( 1 < k < N \) and let \( \Lambda = \{\lambda = (\lambda_1, \ldots, \lambda_k): \lambda_i \) are integers and \( 1 < \lambda_1, \cdots < \lambda_k < N \} \). For any \( \lambda \in \Lambda \) we set \( S(\lambda) = \text{span} \{u_{\lambda_i}\}_{i=1}^k \). Finally, \( d_p(\lambda) \) will denote the \( L_p(a, b) \) distance of \( u_N \) from \( S(\lambda) \) (i.e. \( d_p(\lambda) = \inf \{\|u_N - s\|_p: s \in S(\lambda)\} \)).

The next theorem was proved in [1] for \( p = \infty \) under slightly more restrictive hypotheses and for \( 1 < p < \infty \) by Pinkus [5]. We present a different proof using Theorem 1.

**Theorem 3.** Let \( k \) and \( N \) be as above and \( \lambda \in \Lambda \). Set \( \lambda^* = (N - k, \ldots, N - 1, \) \( \lambda^* \) is a subset of \( \Lambda \) and is defined as follows: if \( \lambda = (\lambda_1, \ldots, \lambda_k) \in \Lambda \), then \( \lambda^* = (N - \lambda_1 - 1, \ldots, N - 1) \).

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\[ d_p(\lambda^*) < d_p(\lambda) \]

for \( 1 < p < \infty \).

**Proof.** Let \( s_\lambda \) be the best \( L_p \) approximation to \( u_N \) from \( S(\lambda) \) with \( \lambda \neq \lambda^* \). Then \( q = u_N - s_\lambda \) has \( k \) distinct zeroes in \( (a, b) \), say \( x_1, \ldots, x_k \) \[4\]. Determine \( s^* \in S(\lambda^*) \) by the equations \( (u_N - s^*)(x_i) = 0 \) for \( i = 1, \ldots, k \). These equations are uniquely solvable by (2.1). Theorem 1 now implies that

\[ |(u_N - s^*)(x)| < |(u_N - s)(x)| \]

for all \( x \in (a, b), x \neq x_i, i = 1, \ldots, k \). Thus for \( 1 < p < \infty \), \( ||u_N - s^*||_p < ||u_N - s||_p \). If \( p = \infty \) we have \( ||u_N - s^*||_\infty < ||u_N - s||_\infty \), but the additional assumption that the \( \{u_i\}_{i=1}^N \) are a Descartes system on \([a, b]\) then yields the strict inequality.

We may obtain a similar result by using Theorem 2 as follows. Let \( \alpha \) be an integer between 1 and \( N \), and let \( l, m, \) and \( k \) be given nonnegative integers satisfying \( 1 < \alpha - l, \alpha + m < N, \) and \( l + m = k \). We set \( \Lambda(l, m; \alpha) = \{\lambda = (\lambda_1, \ldots, \lambda_k): 1 < \lambda_1 < \cdots < \lambda_l < \alpha < \lambda_{l+1} < \cdots < \lambda_k < N, \lambda_i \text{ integers}\} \). For \( \lambda \) and \( \mu \) in \( \Lambda(l, m; \alpha) \) we say \( \lambda \leq \mu \) provided

(i) \( \lambda_i < \mu_i, i = 1, \ldots, l \), and

(ii) \( \lambda_i > \mu_i, i = l + 1, \ldots, k \).

Thus the “largest” element in \( \Lambda(l, m; \alpha) \) is \( \lambda^{**} = (\alpha - l, \alpha - l + 1, \ldots, \alpha - 1, \alpha + 1, \ldots, \alpha + m) \). With this notation we can now state

**Theorem 4.** Let \( \lambda \in \Lambda(l, m; \alpha) \) with \( \lambda \neq \lambda^{**} \), then \( d_p(\lambda^{**}) < d_p(\lambda) \) for \( 1 < p < \infty \).

This theorem is proved in a manner analogous to Theorem 3.

The conjecture of Lorentz is a corollary of Theorem 3 since \( \{x^i\}_{i=0}^N \) is a Descartes system on \((0, \infty)\). More generally, if one wants to approximate \( x^\alpha \), \( 1 < \alpha < N \) with \( \alpha \) an integer, on \( 0 < a < b < \infty \) in the \( L_p[a, b] \) norm by linear combinations of the form \( \sum_{i=1}^k a_i x^{\lambda_i} \) where \( k \) is fixed, \( \lambda_i \neq \alpha, i = 1, \ldots, k \) and \( \lambda_i \) nonnegative integers, then Theorem 4 tells us that the optimal set \( \{\lambda^*_i\}_{i=1}^k \equiv B \) must satisfy \( \{B \cup \alpha\} \) is a set of consecutive integers.

We remark in closing that Theorems 3 and 4 could be strengthened to include approximation in \( L_p(\mu) \) where \( \mu \) is a positive measure such that span \( \{u_i\}_{i=1}^N \) is of dimension \( N \) in \( L_p(\mu) \). This is easy to see since Theorems 1 and 2 are pointwise results.

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