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A NOTE ON VOLTERRA EQUATIONS IN A HILBERT SPACE

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ABSTRACT. An existence and uniqueness result is given on the nonlinear Volterra equation

$$u(t) + (a * Au)(t) \ni f(t), \qquad t \in R^+,$$

when considered in a real Hilbert space. The result extends earlier work by Barbu and Londen.

1. Introduction. In this note we consider the nonlinear Volterra equation

$$u(t) + (a * Au)(t) \ni f(t), \qquad t \in R^+ = [0, \infty), \tag{1.1}$$

where a, A, f are given, $(a * Au)(t) = \int_0^t a(t-\tau)Au(\tau)\,d\tau$, and u is the unknown taking values in a real Hilbert space H. The kernel $a(t)$ is real-valued whereas f maps R^+ into H. The mapping A is assumed to be the subdifferential of a convex, lower semicontinuous function φ: $H \to (-\infty, \infty]$ and is hence maximal monotone. We obtain an existence and uniqueness result on (1.1) which extends earlier work by Barbu and Londen.

A solution of (1.1) on $[0, T]$ is a function $u(t)$ complying with the requirements

$$u \in L^2(0, T; H), \tag{1.2}$$

$$u(t) \in D(A), \quad a.e.\ on\ (0, T), \tag{1.3}$$

and such that there exists $w(t)$ satisfying

$$w \in L^2(0, T; H); \qquad w(t) \in Au(t) \quad a.e.\ on\ (0, T), \tag{1.4}$$

$$u(t) + (a * w)(t) = f(t), \qquad 0 \leqslant t \leqslant T. \tag{1.5}$$

A solution of (1.1) on R^+ is a function u satisfying (1.2)–(1.5) for every $T < \infty$.

By $W^{1,2}_{\mathrm{loc}}(R^+; H)$ we denote the set $\{u | u \in L^2_{\mathrm{loc}}(R^+;\ H),\ du/dt \in L^2_{\mathrm{loc}}(R^+;\ H)\}$, where du/dt is the distributional derivative, and by $BV[0, T]$ the class of real-valued functions defined and of bounded variation on $[0, T]$.

Our result is the following

THEOREM. *Let*

$$A = \partial\varphi, \tag{1.6}$$

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where φ is a proper, convex, lower semicontinuous function mapping H into $(-\infty, \infty]$. Assume

$$a(t) \text{ is locally absolutely continuous on } R^+, \tag{1.7}$$

$$a(0) > 0, \tag{1.8}$$

and suppose there exist α, T_0 such that

$$\omega \int_0^T \sin(\omega t) a'(t)\, dt \leqslant \alpha, \qquad \omega \in R^+, T \leqslant T_0. \tag{1.9}$$

Finally let

$$f \in W^{1,2}_{\mathrm{loc}}(R^+; H), \tag{1.10}$$

$$f(0) \in D(\varphi). \tag{1.11}$$

Then there exists a unique solution of (1.1) *on R^+.*

The key condition is (1.9) which may be formulated as follows. For $T > 0$ define the function $c_T(t)$ by $c_T(t) = a'(t)$, $0 < t < T$, $c_T(t) = 0$, $t > T$, $c_T(t) = -c_T(-t)$, $t < 0$, and assume that there exist positive constants α, T_0 such that if $T \leqslant T_0$ then the Fourier-transform $\hat{c}_T(\omega)$ satisfies the one-sided condition $\omega \hat{c}_T(\omega) \leqslant \alpha$, $\omega \in R$. Observe that the same bound α is required to be good for all $T \in (0, T_0]$. (This condition may in fact be weakened to hold only for some $T_n \downarrow 0$, $n \to \infty$.)

The above result but with the assumption (1.9) strengthened to $a' \in BV[0, T]$, for some $T > 0$, was earlier obtained by Londen, [**4**, Theorem 1], by using partly the same methods of proof as we use here.

If the hypothesis (1.9) is replaced by the requirement that $a(t)$ is of positive type for small arguments, that is by

$$\int_0^t \langle g(\tau), (a * g)(\tau) \rangle \, d\tau \geqslant 0, \qquad 0 \leqslant t \leqslant T, \forall g \in L^2(0, T; H), \tag{1.12}$$

for some $T > 0$, then the existence part of Theorem 1 still follows. This observation has been made and exploited by Barbu [**2**], and is also evident in the following manner from the proof in §2 below. Combine (1.12), (2.6) with the fact, see [**1**, p. 42], that if A is a maximal monotone mapping in H, if $[u_n, v_n] \in A$, if $u_n \rightharpoonup u$, $v_n \rightharpoonup v$, for $n \to \infty$, and if $\limsup_{n,m\to\infty} \langle u_n - u_m, v_n - v_m \rangle \leqslant 0$, then $[u, v] \in A$ and $\langle u_n, v_n \rangle \to \langle u, v \rangle$.

From the remarks above one is led to the following problem (to which we are not aware of any easy solution): characterize the functions $a'(t)$ of unbounded variation on $[0, T]$ that satisfy (1.9) and are such that the corresponding kernel $a(t)$ is not of positive type. As an example showing that such functions do exist we take

$$a'(t) = t \sin(t^{-1}), \qquad 0 < t \leqslant T. \tag{1.13}$$

Clearly $a' \notin BV[0, T]$ and one also has that

$$a(t) \overset{\text{def}}{=} 1 + \int_0^t a'(\tau)\, d\tau$$

does not satisfy (1.12) for any $T > 0$. (This is a consequence of the fact that there exist $t_n \downarrow 0$ such that $a(t_n) > a(0) > 0$. No sufficiently smooth kernel having this property satisfies (1.12).) To see that (1.9) holds we begin by noticing that

$$\omega \int_0^T t \sin(\omega t) \sin(t^{-1})\, dt = \frac{\omega}{2} \int_0^T t \cos(t^{-1} - \omega t)\, dt - \frac{\omega}{2} \int_0^T t \cos(t^{-1} + \omega t)\, dt. \tag{1.14}$$

Introduce the new variable $u = t^{-1} - \omega t$ in the first integral ($= I_1$) on the right side of (1.14) and observe that the resulting integrand (for each fixed ω) equals $\cos u$ multiplied by a bounded monotone function of u. Exploiting this observation one gets $|I_1| \leq T$ which clearly is independent of $\omega \in R^+$. Split the last integral in (1.14) ($= I_2$) in parts as follows:

$$I_2 = \left(\int_0^{2/\sqrt{\omega}} + \int_{2/\sqrt{\omega}}^T \right) \left\{ -\frac{\omega t}{2} \cos(t^{-1} + \omega t) \right\} dt. \tag{1.15}$$

The first integral on the right side of (1.15) is less than 1. In the second integral, introduce the new variable $v = t^{-1} + \omega t$ and make the same monotonicity observation as made above for I_1. Obvious final estimates now give (1.9).

The case when A is a maximal monotone mapping but does not necessarily satisfy (1.6) has been considered by Gripenberg [3].

2. Proof of the Theorem. Let, for $\lambda > 0$, $u_\lambda(t)$, $t \in R^+$, be the solution of

$$u_\lambda(t) + (a * A_\lambda u_\lambda)(t) = f(t), \tag{2.1}$$

where A_λ denotes the Yosida-approximation of A, that is $A_\lambda = \lambda^{-1}[I - J_\lambda]$ with $J_\lambda = [I + \lambda A]^{-1}$. Using well-known techniques, see e.g. the proof of Theorem 1 in [4], and the assumptions (1.6)–(1.8), (1.10), (1.11) one obtains

$$\sup_{\lambda > 0} |u_\lambda|_{L^\infty(0,T;H)} < \infty, \tag{2.2}$$

$$\sup_{\lambda > 0} |A_\lambda u_\lambda|_{L^2(0,T;H)} < \infty, \tag{2.3}$$

for any finite T. Thus there exists $u \in L^\infty_{\mathrm{loc}}(R^+; H)$, $w \in L^2_{\mathrm{loc}}(R^+; H)$ and a sequence $\lambda_n \downarrow 0$ such that

$$u_{\lambda_n} \rightharpoonup u \quad \textit{weak* in } L^\infty_{\mathrm{loc}}(R^+; H), \tag{2.4}$$

$$A_{\lambda_n} u_{\lambda_n} \rightharpoonup w \quad \textit{weakly in } L^2_{\mathrm{loc}}(R^+; H). \tag{2.5}$$

Take any such u, w.

From (2.1) one deduces that for $t \in R^+$

$$\int_0^t \langle g_{\lambda\mu}(\tau), u_\lambda(\tau) - u_\mu(\tau) + (a * g_{\lambda\mu})(\tau) \rangle\, d\tau = 0, \tag{2.6}$$

where $g_{\lambda\mu} = A_\lambda u_\lambda - A_\mu u_\mu$. By the monotonicity of A and the fact that $A_\lambda u_\lambda \in A(J_\lambda u_\lambda)$,

$$\int_0^t \langle g_{\lambda\mu}(\tau), -J_\lambda u_\lambda(\tau) + J_\mu u_\mu(\tau)\rangle \, d\tau \leqslant 0, \qquad t \in R^+. \tag{2.7}$$

Adding (2.6), (2.7), recalling that $u_\lambda - J_\lambda u_\lambda = \lambda A_\lambda u_\lambda$, and using (2.3) one arrives at

$$\limsup_{\lambda,\mu\downarrow 0} \sup_{0 \leqslant t \leqslant T} \int_0^t \langle g_{\lambda\mu}(\tau), (a * g_{\lambda\mu})(\tau)\rangle \, d\tau \leqslant 0, \tag{2.8}$$

where T is any positive number. To proceed we need the following Lemma which is proved in §3.

Lemma. *Let the conditions* (1.7)–(1.9) *hold. Then there exist constants* β, $T > 0$ *such that for any* $g \in L^2(0, T; H)$ *one has*

$$\sup_{0 \leqslant t \leqslant T} \int_0^t \langle g(\tau), (a * g)(\tau)\rangle \, d\tau \geqslant \beta \sup_{0 \leqslant t \leqslant T} \left| \int_0^t g(s)\, ds \right|_H^2 .$$

Combining (2.3) (2.8) with the Lemma, where we take $g = g_{\lambda\mu}$, we conclude that

$$\lim_{\lambda,\mu\downarrow 0} \sup_{0 \leqslant t \leqslant T} \left| \int_0^t g_{\lambda\mu}(s)\, ds \right|_H = 0, \tag{2.9}$$

where T is the constant which the Lemma yields. From (1.7), (2.1) follows

$$u_\lambda(t) - u_\mu(t) + a(0)\int_0^t g_{\lambda\mu}(s)\, ds + \int_0^t a'(t-\tau)\int_0^\tau g_{\lambda\mu}(s)\, ds\, d\tau = 0, \tag{2.10}$$

and hence, after using (2.9) in (2.10) and recalling (2.4), one deduces that

$$\lim_{\lambda\downarrow 0} \sup_{0 \leqslant t \leqslant T} |u_\lambda(t) - u(t)|_H = 0. \tag{2.11}$$

But (2.5), (2.11), and the demicontinuity of A imply

$$u(t) \in D(A), \quad w(t) \in Au(t), \quad \textit{a.e. on } (0, T). \tag{2.12}$$

We now take limits in (2.1) and simultaneously invoke (1.7), (2.5), (2.11), and (2.12). This yields that $u(t)$ is a solution of (1.1) on $[0, T]$.

To obtain a global result we observe at first that $u(t)$ is absolutely continuous on $[0, T]$ and that

$$u'(t) + a(0)w(t) + (a' * w)(t) = f'(t), \tag{2.13}$$

a.e. on this interval. Form the scalar product of $w(t)$ and (2.13) in $L^2(0, T; H)$ and make obvious estimates. From these one draws the conclusion that $u(T) \in D(\varphi)$ which together with (1.7), (1.10) and the fact that $w \in L^2(0, T; H)$ permits us to apply the usual translation-induction arguments providing global existence.

To obtain uniqueness, let u_1, u_2 satisfy (1.1) on $[0, T]$. By the monotonicity of A one then has

$$\int_0^t \langle w_1(\tau) - w_2(\tau), (a * [w_1 - w_2])(\tau)\rangle \, d\tau \leqslant 0,$$

which when combined with the Lemma, where we take $g = w_1 - w_2$, yields

$\int_0^t [w_1(\tau) - w_2(\tau)]\, d\tau = 0$, $t \in [0, T]$. Keeping this in mind it is not difficult to show that $u_1 \equiv u_2$, $t \in [0, T]$. Again the usual translation-induction arguments supply uniqueness on R^+.

3. Proof of the Lemma. Take any $T > 0$, $T \leqslant T_0$ satisfying (possible by (1.7), (1.8))

$$a(T) \geqslant 32\left[\alpha T + \int_0^T |a'(s)|\, ds\right]. \tag{3.1}$$

Then choose any $g \in L^2(0, T; H)$ and an arbitrary $t \in (0, T]$. Define b, φ by

$$\begin{gathered} b(\tau) = a(\tau) - a(T), \quad \tau \in [0, T], \qquad b(\tau) = 0, \quad \tau > T, \\ b(-\tau) = b(\tau), \qquad \tau \in R^+, \end{gathered} \tag{3.2}$$

$$\varphi(\tau) = g(\tau) - t^{-1}\int_0^t g(s)\, ds, \quad \tau \in [0, t]; \quad \varphi(\tau) = 0, \quad \tau \notin [0, t], \tag{3.3}$$

and consider

$$\Phi \stackrel{\text{def}}{=} \frac{1}{2} \int_{-\infty}^{\infty} \left\langle \varphi(\tau), \int_{-\infty}^{\infty} b(\tau - s)\varphi(s)\, ds \right\rangle d\tau. \tag{3.4}$$

Now use (3.3) and the fact that b is even in (3.4), then perform an integration by parts; permissible by (1.7), (3.2); make obvious estimates and use (3.1). This results in

$$\Phi - \int_0^t \langle g(\tau), (b * g)(\tau)\rangle\, d\tau \leqslant 8^{-1} a(T) \sup_{0 \leqslant \tau \leqslant t} \left| \int_0^\tau g(s)\, ds \right|_H^2. \tag{3.5}$$

But (H_c denotes the complexification of H and $\hat{\varphi}$, $\hat{b}$ the Fourier transforms of φ, b)

$$\begin{aligned} \Phi &= \frac{1}{4\pi} \int_{-\infty}^{\infty} |\hat{\varphi}(\omega)|_{H_c}^2 \hat{b}(\omega)\, d\omega \geqslant -\frac{a(T)}{64\pi T} \int_{-\infty}^{\infty} \left| \frac{\hat{\varphi}(\omega)}{\omega} \right|_{H_c}^2 d\omega \\ &= -\frac{a(T)}{32T} \int_0^t \left| \int_0^\tau \varphi(s)\, ds \right|_H^2 d\tau \geqslant -\frac{a(T)}{8} \sup_{0 \leqslant \tau \leqslant t} \left| \int_0^\tau g(s)\, ds \right|_H^2 \end{aligned} \tag{3.6}$$

where the first inequality follows by (1.9) and (3.1) and where the second is evident from (3.3). On the other hand (3.5), (3.6) and the definition of b show that

$$\begin{aligned} &\int_0^t \langle g(\tau), (a * g)(\tau)\rangle\, d\tau \\ &\qquad = 2^{-1} a(T) \left| \int_0^t g(s)\, ds \right|_H^2 + \int_0^t \langle g(\tau), (b * g)(\tau)\rangle\, d\tau \\ &\qquad \geqslant 2\beta \left| \int_0^t g(s)\, ds \right|_H^2 - \beta \sup_{0 \leqslant \tau \leqslant t} \left| \int_0^\tau g(s)\, ds \right|_H^2, \end{aligned} \tag{3.7}$$

where $\beta = 2^{-2} a(T)$. Taking the supremum over $[0, T]$ of each side in (3.7) one gets the desired conclusion.

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