A NOTE ON VOLterra equations
IN A Hilbert SPACE

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Abstract. An existence and uniqueness result is given on the nonlinear
Volterra equation
\[ u(t) + (a \ast Au)(t) \equiv f(t), \quad t \in R^+, \]
when considered in a real Hilbert space. The result extends earlier work by
Barbu and Londen.

1. Introduction. In this note we consider the nonlinear Volterra equation
\[ u(t) + (a \ast Au)(t) \equiv f(t), \quad t \in R^+ = [0, \infty), \quad (1.1) \]
where \( a, A, f \) are given, \( (a \ast Au)(t) = \int_0^t a(t - \tau)Au(\tau) \, d\tau \), and \( u \) is the
unknown taking values in a real Hilbert space \( H \). The kernel \( a(t) \) is real-val-
ued whereas \( f \) maps \( R^+ \) into \( H \). The mapping \( A \) is assumed to be the
subdifferential of a convex, lower semicontinuous function \( \varphi : H \to (-\infty, \infty] \)
and is hence maximal monotone. We obtain an existence and uniqueness
result on (1.1) which extends earlier work by Barbu and Londen.

A solution of (1.1) on \([0, T]\) is a function \( u(t) \) complying with the require-
ments
\[ u \in L^2(0, T; H), \quad (1.2) \]
\[ u(t) \in D(A), \quad a.e. \, on \, (0, T), \quad (1.3) \]
and such that there exists \( w(t) \) satisfying
\[ w \in L^2(0, T; H); \quad w(t) \in Au(t) \quad a.e. \, on \, (0, T), \quad (1.4) \]
\[ u(t) + (a \ast w)(t) = f(t), \quad 0 < t < T. \quad (1.5) \]
A solution of (1.1) on \( R^+ \) is a function \( u \) satisfying (1.2)–(1.5) for every
\( T < \infty \).

By \( W^{1,2}_{\text{loc}}(R^+; H) \) we denote the set \( \{ u \in L^2_{\text{loc}}(R^+; H), \, du/dt \in L^2_{\text{loc}}(R^+; H) \} \), where \( du/dt \) is the distributional derivative, and by \( BV[0, T] \)
the class of real-valued functions defined and of bounded variation on \([0, T]\).

Our result is the following

Theorem. Let
\[ A = \partial \varphi, \quad (1.6) \]
where \( \varphi \) is a proper, convex, lower semicontinuous function mapping \( H \) into \( (-\infty, \infty) \). Assume

\[
a(t) \text{ is locally absolutely continuous on } R^+, \\
a(0) > 0,
\]

and suppose there exist \( \alpha, T_0 \) such that

\[
\omega \int_0^T \sin(\omega t)a'(t) \, dt < \alpha, \quad \omega \in R^+, \, T < T_0.
\]

Finally let

\[
f \in W^{1,2}_{\text{loc}}(R^+; H), \\
f(0) \in D(\varphi).
\]

Then there exists a unique solution of \((\text{1.1})\) on \( R^+ \).

The key condition is (1.9) which may be formulated as follows. For \( T > 0 \) define the function \( c_T(t) \) by \( c_T(t) = a'(t), \, 0 < t < T, \, c_T(t) = 0, \, t > T, \), \( c_T(t) = -c_T(-t), \, t < 0, \) and assume that there exist positive constants \( \alpha, T_0 \) such that if \( T < T_0 \) then the Fourier-transform \( \hat{c}_T(\omega) \) satisfies the one-sided condition \( \omega \hat{c}_T(\omega) < \alpha, \, \omega \in R \). Observe that the same bound \( \alpha \) is required to be good for all \( T \in (0, T_0] \). (This condition may in fact be weakened to hold only for some \( T_n \downarrow 0, \, n \to \infty \).)

The above result but with the assumption (1.9) strengthened to \( a' \in BV[0, T] \), for some \( T > 0 \), was earlier obtained by Londen, [4, Theorem 1], by using partly the same methods of proof as we use here.

If the hypothesis (1.9) is replaced by the requirement that \( a(t) \) is of positive type for small arguments, that is by

\[
\int_0^t \langle g(\tau), (a * g)(\tau) \rangle \, d\tau > 0, \quad 0 < t < T, \forall g \in L^2(0, T; H), \tag{1.12}
\]

for some \( T > 0 \), then the existence part of Theorem 1 still follows. This observation has been made and exploited by Barbu [2], and is also evident in the following manner from the proof in §2 below. Combine (1.12), (2.6) with the fact, see [1, p. 42], that if \( A \) is a maximal monotone mapping in \( H \), if \( [u_n, v_n] \in A \), if \( u_n \to u, \, v_n \to v, \) for \( n \to \infty \), and if \( \limsup_{n,m \to \infty} \langle u_n - u_m, v_n - v_m \rangle < 0 \), then \( [u, v] \in A \) and \( \langle u_n, v_n \rangle \to \langle u, v \rangle \).

From the remarks above one is led to the following problem (to which we are not aware of any easy solution): characterize the functions \( a'(t) \) of unbounded variation on \([0, T]\) that satisfy (1.9) and are such that the corresponding kernel \( a(t) \) is not of positive type. As an example showing that such functions do exist we take

\[
a'(t) = t \sin(t^{-1}), \quad 0 < t < T. \tag{1.13}
\]

Clearly \( a' \notin BV[0, T] \) and one also has that

\[
a(t) \overset{\text{def}}{=} 1 + \int_0^t a'(\tau) \, d\tau
\]

does not satisfy (1.12) for any \( T > 0 \). (This is a consequence of the fact that
there exist \( t_n \downarrow 0 \) such that \( a(t_n) > a(0) > 0. \) No sufficiently smooth kernel having this property satisfies (1.12). To see that (1.9) holds we begin by noticing that

\[
\omega \int_0^T t \sin(\omega t) \sin(t^{-1}) \, dt = \frac{\omega}{2} \int_0^T t \cos(t^{-1} - \omega t) \, dt - \frac{\omega}{2} \int_0^T t \cos(t^{-1} + \omega t) \, dt.
\]

(1.14)

Introduce the new variable \( u = t^{-1} - \omega t \) in the first integral (= \( I_1 \)) on the right side of (1.14) and observe that the resulting integrand (for each fixed \( \omega \)) equals \( \cos u \) multiplied by a bounded monotone function of \( u. \) Exploiting this observation one gets \( |I_1| \lesssim \tau \) which clearly is independent of \( \omega \in \mathbb{R}^+. \) Split the last integral in (1.14) (= \( I_2 \)) in parts as follows:

\[
I_2 = \left( \int_0^{2/\sqrt{\omega}} + \int_{2/\sqrt{\omega}}^T \right) \left\{ - \frac{\omega t}{2} \cos(t^{-1} + \omega t) \right\} \, dt. \]

(1.15)

The first integral on the right side of (1.15) is less than 1. In the second integral, introduce the new variable \( v = t^{-1} + \omega t \) and make the same monotonicity observation as made above for \( I_1. \) Obvious final estimates now give (1.9).

The case when \( A \) is a maximal monotone mapping but does not necessarily satisfy (1.6) has been considered by Gripenberg [3].

2. Proof of the Theorem. Let, for \( \lambda > 0, u_\lambda(t), t \in \mathbb{R}^+, \) be the solution of

\[
u_\lambda(t) + (a \ast A_\lambda u_\lambda)(t) = f(t), \quad (2.1)
\]

where \( A_\lambda \) denotes the Yosida-approximation of \( A, \) that is \( A_\lambda = \lambda^{-1}[I - J_{\lambda}] \) with \( J_{\lambda} = [I + \lambda A]^{-1}. \) Using well-known techniques, see e.g. the proof of Theorem 1 in [4], and the assumptions (1.6)–(1.8), (1.10), (1.11) one obtains

\[
\sup_{\lambda > 0} |u_\lambda|_{L^\infty(0,T;H)} < \infty, \quad (2.2)
\]

\[
\sup_{\lambda > 0} |A_\lambda u_\lambda|_{L^2(0,T;H)} < \infty, \quad (2.3)
\]

for any finite \( T. \) Thus there exists \( u \in L^\infty_{\text{loc}}(\mathbb{R}^+; H), w \in L^2_{\text{loc}}(\mathbb{R}^+; H) \) and a sequence \( \lambda_n \downarrow 0 \) such that

\[
u_\lambda \rightharpoonup u \quad \text{weak* in } L^\infty_{\text{loc}}(\mathbb{R}^+; H), \quad (2.4)
\]

\[
A_\lambda u_\lambda \rightharpoonup w \quad \text{weakly in } L^2_{\text{loc}}(\mathbb{R}^+; H). \quad (2.5)
\]

Take any such \( u, w. \)

From (2.1) one deduces that for \( t \in \mathbb{R}^+

\[
\int_0^t \left\langle g_\lambda(\tau), u_\lambda(\tau) - u_\mu(\tau) + (a \ast g_\lambda)(\tau) \right\rangle \, d\tau = 0, \quad (2.6)
\]

where \( g_\lambda = A_\lambda u_\lambda - A_\mu u_\mu. \) By the monotonicity of \( A \) and the fact that \( A_\lambda u_\lambda \in A(J_{\lambda} u_\lambda), \)
\[
\int_0^t \langle g_{\lambda}(\tau), -J_\lambda u_\lambda(\tau) + J_\mu u_\mu(\tau) \rangle \, d\tau < 0, \quad t \in R^+.
\] (2.7)

Adding (2.6), (2.7), recalling that \( u_\lambda - J_\lambda u_\lambda = \lambda A_\lambda u_\lambda \), and using (2.3) one arrives at

\[
\limsup_{\lambda, \mu \downarrow 0} \sup_{0 < t < T} \int_0^t \langle g_{\lambda}(\tau), (a \ast g_{\mu})(\tau) \rangle \, d\tau < 0,
\] (2.8)

where \( T \) is any positive number. To proceed we need the following Lemma which is proved in §3.

**Lemma.** Let the conditions (1.7)—(1.9) hold. Then there exist constants \( \beta, T > 0 \) such that for any \( g \in L^2(0,T;H) \) one has

\[
\sup_{0 < t < T} \int_0^t \langle g(\tau), (a \ast g)(\tau) \rangle \, d\tau > \beta \sup_{0 < t < T} \left| \int_0^t g(s) \, ds \right|^2_H.
\]

Combining (2.3) (2.8) with the Lemma, where we take \( g = g_{\lambda_\mu} \), we conclude that

\[
\lim_{\lambda, \mu \downarrow 0} \sup_{0 < t < T} \left| \int_0^t g_{\lambda_\mu}(s) \, ds \right|^2_H = 0,
\] (2.9)

where \( T \) is the constant which the Lemma yields. From (1.7), (2.1) follows

\[
u(0) - \nu(0) + a(0) \int_0^t g_{\lambda_\mu}(s) \, ds + \int_0^t \alpha(t - \tau) \int_0^\tau g_{\lambda_\mu}(s) \, ds \, d\tau = 0,
\] (2.10)

and hence, after using (2.9) in (2.10) and recalling (2.4), one deduces that

\[
\lim_{\lambda_\mu \downarrow 0} \sup_{0 < t < T} |u(t) - u(t)|_H = 0.
\] (2.11)

But (2.5), (2.11), and the demicontinuity of \( A \) imply

\[
u(t) \in D(A), \quad w(t) \in Au(t), \quad a.e. \, on \, (0,T).
\] (2.12)

We now take limits in (2.1) and simultaneously invoke (1.7), (2.5), (2.11), and (2.12). This yields that \( u(t) \) is a solution of (1.1) on \([0,T]\).

To obtain a global result we observe at first that \( u(t) \) is absolutely continuous on \([0, T]\) and that

\[
u'(t) + a(0)w(t) + (a' \ast w)(t) = f'(t),
\] (2.13)
a.e. on this interval. Form the scalar product of \( w(t) \) and (2.13) in \( L^2(0,T;H) \) and make obvious estimates. From these one draws the conclusion that \( u(T) \in D(\varphi) \) which together with (1.7), (1.10) and the fact that \( w \in L^2(0,T;H) \) permits us to apply the usual translation-induction arguments providing global existence.

To obtain uniqueness, let \( u_1, u_2 \) satisfy (1.1) on \([0,T]\). By the monotonicity of \( A \) one then has

\[
\int_0^t \langle w_1(\tau) - w_2(\tau), (a \ast [w_1 - w_2])(\tau) \rangle \, d\tau < 0,
\]

which when combined with the Lemma, where we take \( g = w_1 - w_2 \), yields
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\[ f_0[w_1(\tau) - w_2(\tau)] \, d\tau = 0, \quad \tau \in [0, T]. \]
Keeping this in mind it is not difficult to show that \( u_1 \equiv u_2, \quad \tau \in [0, T]. \) Again the usual translation-induction arguments supply uniqueness on \( R^+. \)

3. Proof of the Lemma. Take any \( T > 0, \quad T < T_0 \) satisfying (possible by (1.7), (1.8))
\[
a(T) > 32 \left[ aT + \int_0^T |a'(s)| \, ds \right]. \tag{3.1}
\]
Then choose any \( g \in L^2(0, T; H) \) and an arbitrary \( t \in (0, T). \) Define \( b, \varphi \) by
\[
b(\tau) = a(\tau) - a(T), \quad \tau \in [0, T], \quad b(\tau) = 0, \quad \tau > T,
\]
\[
b(-\tau) = b(\tau), \quad \tau \in R^+, \tag{3.2}
\]
\[
\varphi(\tau) = g(\tau) - t^{-1} \int_0^\tau g(s) \, ds, \quad \tau \in [0, t]; \quad \varphi(\tau) = 0, \quad \tau \notin [0, t], \tag{3.3}
\]
and consider
\[
\Phi \overset{\text{def}}{=} \frac{1}{2} \int_{-\infty}^{\infty} \left( \varphi(\tau), \int_{-\infty}^{\infty} b(\tau - s)\varphi(s) \, ds \right) d\tau. \tag{3.4}
\]
Now use (3.3) and the fact that \( b \) is even in (3.4), then perform an integration by parts; permissible by (1.7), (3.2); make obvious estimates and use (3.1). This results in
\[
\Phi - \int_0^\tau \langle g(\tau), (b * g)(\tau) \rangle \, d\tau < 8^{-1}a(T) \sup_{0 < \tau < t} \left| \int_0^\tau g(s) \, ds \right|^2. \tag{3.5}
\]
But \( (H_c \text{ denotes the complexification of } H \text{ and } \hat{\varphi}, \hat{b} \text{ the Fourier transforms of } \varphi, b) \)
\[
\Phi = \frac{1}{4\pi} \int_{-\infty}^{\infty} \left| \hat{\varphi}(\omega) \right|^2 \left| \hat{b}(\omega) \right|^2 \, d\omega > -\frac{a(T)}{64\pi T} \int_{-\infty}^{\infty} \left| \hat{\varphi}(\omega) \right|^2 \left| \hat{b}(\omega) \right|^2 \, d\omega \tag{3.6}
\]
\[
= -\frac{a(T)}{32T} \int_0^\tau \left| \varphi(s) \right|^2 \, ds - \frac{a(T)}{8} \sup_{0 < \tau < t} \left| \int_0^\tau g(s) \, ds \right|^2 \tag{3.6}
\]
where the first inequality follows by (1.9) and (3.1) and where the second is evident from (3.3). On the other hand (3.5), (3.6) and the definition of \( b \) show that
\[
\int_0^\tau \langle g(\tau), (a * g)(\tau) \rangle \, d\tau
\]
\[
= 2^{-1}a(T) \left| \int_0^\tau g(s) \, ds \right|^2 + \int_0^\tau \langle g(\tau), (b * g)(\tau) \rangle \, d\tau \tag{3.7}
\]
\[
> 2\beta \left| \int_0^\tau g(s) \, ds \right|^2 - \beta \sup_{0 < \tau < t} \left| \int_0^\tau g(s) \, ds \right|^2,
\]
where \( \beta = 2^{-2}a(T). \) Taking the supremum over \([0, T]\) of each side in (3.7) one gets the desired conclusion.
REFERENCES


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