A MOMENT PROBLEM ON JORDAN DOMAINS

MAKOTO SAKAI

ABSTRACT. Let $D_1$, $D_2$ be Jordan domains on the complex $z$-plane such that $\int_{D_1} z^n \, dm = \int_{D_2} z^n \, dm$ for every nonnegative integer $n$. Here $m$ denotes two-dimensional Lebesgue measure. Does it follow that $D_1 = D_2$? This moment problem on Jordan domains was posed by H. S. Shapiro [2, p. 193, Problem 1]. In this paper we construct a counterexample and study conditions on $D_1$ and $D_2$ which imply that the above equality does not hold for some $n$.

1. A counterexample. We first construct a counterexample. Let $R_1 = \{z|3 < |z + 1| < \sqrt{10}\}$, $R_2 = \{z|3 < |z - 1| < \sqrt{10}\}$, $E = R_1 \cap R_2 \cap \{z|\text{Im } z > 0\}$ and $F = (\Delta_3(-1) \cap \Delta_3(1)) - (\Delta_1(-1) \cup \Delta_1(1)) \cap \{z|\text{Im } z > 0\}$, where $\Delta_r(c)$ denotes the open disc with radius $r$ and center at $c$. Set $D_1 = (R_1 \cup \Delta_1(1) \cup \bar{F})^c - \bar{E}$ and $D_2 = (R_2 \cup \Delta_1(-1) \cup \bar{F})^c - \bar{E}$, where $\bar{F}$ denotes the closure of $F$ and $A^c$ denotes the interior of $A$ (Figure 1, $D_2$ is the reflection of $D_1$ in the imaginary axis). Since

$$\int_{R_1 \cup \Delta_1(1)} z^n \, dm = \pi \{(-1)^n + 1^n\} = \int_{R_2 \cup \Delta_1(-1)} z^n \, dm,$$

two distinct Jordan domains $D_1$ and $D_2$ satisfy

$$\int_{D_1} z^n \, dm = \int_{D_2} z^n \, dm$$

for every nonnegative integer $n$.

![Figure 1](image-url)
Remark. By deforming $F$ in the above counterexample we can construct another counterexample with $D_1$ and $D_2$ which are not congruent.

2. Conditions. We next summarize the known results.

**Proposition 1.** Let $D_1$, $D_2$ be Jordan domains satisfying one of the following conditions:

(i) $D_1$ and $D_2$ are distinct and there is an analytic $L^1$ function $\phi$ on a Jordan domain containing $D_1 \cup D_2$ such that $\text{Re} \phi(z) > 0$ on $D_1 - D_2$ and $\text{Re} \phi(z) < 0$ on $D_2 - D_1$.

(ii) $D_1$ and $D_2$ are disjoint or intersect in just one point.

(iii) We denote by $\gamma$ the boundary of the unbounded complementary component of $D_1 \cup D_2$ and denote by $D_i^\circ$ the exterior of $D_i$. The restriction of $\gamma$ to $D_1^\circ \cup D_2^\circ$ is not analytic.

Then

$$\int_{D_1} z^n \, dm \neq \int_{D_2} z^n \, dm$$

for some nonnegative integer $n$.

Condition (i) is very useful. Condition (ii) was given by H. S. Shapiro [2] (see also [1]) and condition (iii) was given implicitly by D. Aharonov and H. S. Shapiro [1, Lemmas 2.2 and 6.1] (see also P. J. Davis [4, p. 21]).

Finally we give our new condition. To do so, we recall the following proposition proved in [6].

**Proposition A.** Let $D$ be a domain containing the origin 0. Let $\nu$ be an $L^1$ function on $\mathbb{C}$ such that $\nu(z) \geq k$ a.e. on $D$ for a positive number $k$ and $\nu(z) = 0$ a.e. on the complement of $D$. If $f'(0) = \int_D f' \nu \, dm / \int \nu \, dm$ for every analytic function $f$ on $D$ such that $\int_D |f|^2 \nu \, dm < \infty$, then $D \subset \Delta_r(0)$, where $r = (\int \nu \, dm / k\pi)^{1/2}$. The equality $\sup_{z \in D} |z| = r$ holds if and only if $\nu(z) = k$ a.e. on $D$ and $D = \Delta_r(0) - E$, where $E$ is a relatively closed subset of $\Delta_r(0)$.
such that \( E \cap K \) is removable with respect to analytic functions with finite Dirichlet integrals for every compact subset \( K \) of \( \Delta_e(0) \).

By using this proposition we have

**Proposition 2.** Let \( D_1, D_2 \) be two distinct Jordan domains. Suppose there is an open disc \( \Delta = \Delta_e(c) \) having the following properties:
(i) \( D_1 \cup D_2 \subset \Delta \).
(ii) \( \chi_{D_1 \cup (\Delta - D_1)} = \chi_\Omega \) a.e. on \( \mathbb{C} \) for a simply connected domain \( \Omega \) such that
(a) \( c \in \Omega \),
(b) every analytic \( L^2 \) function on \( \Omega \) can be approximated arbitrarily closely in the \( L^2 \) norm by a sequence of polynomials, where \( \chi_\Omega \) denotes the characteristic function of \( \Omega \).

Then
\[
\int_{D_1} z^n \, dm \neq \int_{D_2} z^n \, dm
\]
for some nonnegative integer \( n \).

**Proof.** Assume that \( \int_{D_1} z^n \, dm = \int_{D_2} z^n \, dm \) for every \( n \) and set \( v(z) = \chi_\Omega(z) + \chi_{D_2 - D_1}(z) \). Then
\[
\int_{\Omega} z^n v(z) \, dm = \int_{D_1 \cup (\Delta - D_1)} z^n \, dm + \int_{D_2 \cap (\Delta - D_1)} z^n \, dm = \int_{\Delta} z^n \, dm + \int_{D_2} z^n \, dm - \int_{D_1} z^n \, dm = c^n m(\Delta).
\]

Since every analytic \( L^2 \) function on \( \Omega \) can be approximated by polynomials, we have
\[
f(c) = \int_{\Omega} f v \, dm / \int v \, dm
\]
for every analytic \( L^2 \) function on \( \Omega \). From the definition of \( v(z) \), it follows that \( v(z) > 1 \) a.e. on \( \Omega \), \( v(z) = 0 \) a.e. on the complement of \( \Omega \), \( \int v \, dm = \pi r^2 \) and \( \sup_{z \in \Omega} |z - c| = r \). Hence, by Proposition A, we have \( \chi_{D_2 - D_1}(z) = 0 \) a.e. on \( \mathbb{C} \). Since \( D_1 \) and \( D_2 \) are Jordan domains satisfying \( m(D_1) = m(D_2) \), we have \( D_1 = D_2 \). This is a contradiction.

Proposition 2 is applicable to the case treated in Problem 4 of [2] (Figure 2, cf. Remark). In this case \( \Omega \) is a Jordan domain, and so \( \Omega \) has an approximation property mentioned in (b) of Proposition 2. Carathéodory domains also have this property; \( \Omega \) is called a Carathéodory domain if its boundary coincides with the boundary of the unbounded complementary component of its closure. There are non-Carathéodory domains having this property (see S. N. Mergelyan [5] and J. E. Brennan [3]). Proposition 2 is also applicable to these cases (Figure 3).

**Remark.** If \( \Omega \) is a Carathéodory domain such that \( \Omega^c \) is connected, for example, we can omit (a) of Proposition 2. In fact, for every \( \xi \in \Omega^c \),
\[
1 / (z - \xi) \]
can be approximated uniformly on \( \overline{\Omega} \cup \{ c \} \) by polynomials.
Hence, from (1), we have
\[ \hat{\nu}(\zeta) = \frac{m(\Delta)}{(c - \zeta)}, \]
where \( \hat{\nu} \) denotes the Cauchy transform of \( \nu \). Since \( \hat{\nu} \) is continuous, it follows that \( c \in \Omega \).

REFERENCES


DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, HIROSHIMA UNIVERSITY, HIROSHIMA, JAPAN