A MOMENT PROBLEM ON JORDAN DOMAINS

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ABSTRACT. Let $D_1, D_2$ be Jordan domains on the complex $z$-plane such that $\int_{D_1} z^n \, dm = \int_{D_2} z^n \, dm$ for every nonnegative integer $n$. Here $m$ denotes two-dimensional Lebesgue measure. Does it follow that $D_1 = D_2$? This moment problem on Jordan domains was posed by H. S. Shapiro [2, p. 193, Problem 1]. In this paper we construct a counterexample and study conditions on $D_1$ and $D_2$ which imply that the above equality does not hold for some $n$.

1. A counterexample. We first construct a counterexample. Let $R_1 = \{z | 3 < |z + 1| < \sqrt{10}\}$, $R_2 = \{z | 3 < |z - 1| < \sqrt{10}\}$, $E = R_1 \cap R_2 \cap \{z | \text{Im} \ z > 0\}$ and $F = \{ (\Delta_3(-1) \cap \Delta_3(1)) - (\Delta_1(-1) \cup \Delta_1(1)) \} \cap \{z | \text{Im} \ z > 0\}$, where $\Delta_r(c)$ denotes the open disc with radius $r$ and center at $c$. Set $D_1 = (R_1 \cup \Delta_1(1) \cup \overline{F})^c - E$ and $D_2 = (R_2 \cup \Delta_1(-1) \cup \overline{F})^c - E$, where $\overline{F}$ denotes the closure of $F$ and $A^c$ denotes the interior of $A$ (Figure 1, $D_2$ is the reflection of $D_1$ in the imaginary axis). Since

$$\int_{R_1 \cup \Delta_1(1)} z^n \, dm = \pi \{(-1)^n + 1^n\} = \int_{R_2 \cup \Delta_1(-1)} z^n \, dm,$$

two distinct Jordan domains $D_1$ and $D_2$ satisfy

$$\int_{D_1} z^n \, dm = \int_{D_2} z^n \, dm$$

for every nonnegative integer $n$.

![Graphical Representation of the Domains](https://example.com/graph.png)

FIGURE 1

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Remark. By deforming $F$ in the above counterexample we can construct another counterexample with $D_1$ and $D_2$ which are not congruent.

2. Conditions. We next summarize the known results.

Proposition 1. Let $D_1, D_2$ be Jordan domains satisfying one of the following conditions:

(i) $D_1$ and $D_2$ are distinct and there is an analytic $L^1$ function $\phi$ on a Jordan domain containing $D_1 \cup D_2$ such that $\Re \phi(z) > 0$ on $D_1 - D_2$ and $\Re \phi(z) < 0$ on $D_2 - D_1$.

(ii) $D_1$ and $D_2$ are disjoint or intersect in just one point.

(iii) We denote by $\gamma$ the boundary of the unbounded complementary component of $D_1 \cup D_2$ and denote by $D_1'$ the exterior of $D_1$. The restriction of $\gamma$ to $D_1' \cup D_2'$ is not analytic.

Then

$$\int_{D_1} z^n \ dm \neq \int_{D_2} z^n \ dm$$

for some nonnegative integer $n$.

Condition (i) is very useful. Condition (ii) was given by H. S. Shapiro [2] (see also [1]) and condition (iii) was given implicitly by D. Aharonov and H. S. Shapiro [1, Lemmas 2.2 and 6.1] (see also P. J. Davis [4, p. 21]).

Finally we give our new condition. To do so, we recall the following proposition proved in [6].

Proposition A. Let $D$ be a domain containing the origin 0. Let $v$ be an $L^1$ function on $C$ such that $v(z) > k$ a.e. on $D$ for a positive number $k$ and $v(z) = 0$ a.e. on the complement of $D$. If $f'(0) = \int_D f' v \ dm / \int v \ dm$ for every analytic function $f$ on $D$ such that $\int_D |f'|^2 v \ dm < \infty$, then $D \subset \Delta_r(0)$, where $r = (\int v \ dm / k \pi)^{1/2}$. The equality $\sup_{z \in D} |z| = r$ holds if and only if $v(z) = k$ a.e. on $D$ and $D = \Delta_r(0) - E$, where $E$ is a relatively closed subset of $\Delta_r(0)$.
such that $E \cap K$ is removable with respect to analytic functions with finite Dirichlet integrals for every compact subset $K$ of $\Delta_c(0)$.

By using this proposition we have

**Proposition 2.** Let $D_1$, $D_2$ be two distinct Jordan domains. Suppose there is an open disc $\Delta = \Delta_c(c)$ having the following properties:

(i) $D_1 \cup D_2 \subset \Delta$.

(ii) $\chi_{D_2 \cup \Delta - D_1} = \chi_\Omega$ a.e. on $\mathbb{C}$ for a simply connected domain $\Omega$ such that

(a) $c \in \Omega$,

(b) every analytic $L^2$ function on $\Omega$ can be approximated arbitrarily closely in the $L^2$ norm by a sequence of polynomials, where $\chi_\Omega$ denotes the characteristic function of $\Omega$.

Then

$$\int_{D_1} z^n dm \neq \int_{D_2} z^n dm$$

for some nonnegative integer $n$.

**Proof.** Assume that $\int_{D_1} z^n dm = \int_{D_2} z^n dm$ for every $n$ and set $\nu(z) = \chi_\Omega(z) + \chi_{D_2 - D_1}(z)$. Then

$$\int_{\Omega} z^n \nu(z) dm = \int_{D_2 \cup \Delta - D_1} z^n dm + \int_{D_2 \cap (\Delta - D_1)} z^n dm = \int_{\Delta} z^n dm + \int_{D_2} z^n dm - \int_{D_1} z^n dm = c^n m(\Delta).$$

Since every analytic $L^2$ function on $\Omega$ can be approximated by polynomials, we have

$$f(c) = \int_{\Omega} \nu dm / \int \nu dm$$

for every analytic $L^2$ function on $\Omega$. From the definition of $\nu(z)$, it follows that $\nu(z) > 1$ a.e. on $\Omega$, $\nu(z) = 0$ a.e. on the complement of $\Omega$, $\int \nu dm = \pi r^2$ and $\sup_{z \in \Omega} |z - c| = r$. Hence, by Proposition A, we have $\chi_{D_2 - D_1}(z) = 0$ a.e. on $\mathbb{C}$. Since $D_1$ and $D_2$ are Jordan domains satisfying $m(D_1) = m(D_2)$, we have $D_1 = D_2$. This is a contradiction.

Proposition 2 is applicable to the case treated in Problem 4 of [2] (Figure 2, cf. Remark). In this case $\Omega$ is a Jordan domain, and so $\Omega$ has an approximation property mentioned in (b) of Proposition 2. Carathéodory domains also have this property; $\Omega$ is called a Carathéodory domain if its boundary coincides with the boundary of the unbounded complementary component of its closure. There are non-Carathéodory domains having this property (see S. N. Mergelyan [5] and J. E. Brennan [3]). Proposition 2 is also applicable to these cases (Figure 3).

**Remark.** If $\Omega$ is a Carathéodory domain such that $\Omega^c$ is connected, for example, we can omit (a) of Proposition 2. In fact, for every $\xi \in \Omega^c$, $1/(z - \xi)$ can be approximated uniformly on $\overline{\Omega} \cup \{c\}$ by polynomials.
Hence, from (1), we have
\[ \tilde{\nu}(\zeta) = \frac{m(\Delta)}{(c - \zeta)}, \]
where \( \tilde{\nu} \) denotes the Cauchy transform of \( \nu \). Since \( \tilde{\nu} \) is continuous, it follows that \( c \in \Omega \).

REFERENCES


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