THE FREDHOLM RADIUS OF A BUNDLE
OF CLOSED LINEAR OPERATORS

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ABSTRACT. Given a bundle of linear operators \( T - \lambda S \), where \( T \) is closed and \( S \) is bounded, a sequence \( \{ \delta_m(T : S) \} \) of extended real numbers is defined. If \( T \) is a Fredholm operator, the limit \( \lim \delta_m(T : S)^{1/m} \) exists and is equal to the supremum of all \( r > 0 \) such that \( T - \lambda S \) is a Fredholm operator for \( |\lambda| < r \).

Throughout this paper \( X \) and \( Y \) are complex Banach spaces, \( T \) is a closed linear operator with domain \( D(T) \) in \( X \) and range \( R(T) \) in \( Y \), and \( S \) is a bounded linear operator from \( X \) into \( Y \). \( K(X) \) is the space of compact linear operators on \( X \), and \( \Phi(T : S) \) is the set of those complex numbers \( \lambda \) for which \( T - \lambda S \) is a Fredholm operator.

Given \( m > 1 \), the element \( (x_1, \ldots, x_m) \) of \( D(T)^m \) is called a chain for \( T \) and \( S \) if \( Tx_i = Sx_{i-1} \) for \( i = 2, \ldots, m \). Put

\[
\delta_m = \delta_m(T : S) = \sup_{C \in K(X)} \inf_{(x_1, \ldots, x_m)} \frac{\|Tx_1\|}{\|(I - C)x_m\|},
\]

where the infimum is taken over all chains \( (x_1, \ldots, x_m) \) for \( T \) and \( S \). Here \( I \) denotes the identity mapping in \( X \).

When \( X = Y \) and \( S = I \), the chains for \( T \) and \( S \) are of the form \( (Tm^{-1}x, \ldots, Tx, x) \) with \( x \in D(T^m) \), and

\[
\delta_m(T : I) = \delta_1(T^m : I) = \sup_{C \in K(X)} \inf_{x \in D(T)} \frac{\|T^mx\|}{\|(I - C)x\|}.
\]

Roughly speaking \( \delta_1(T : I) \) is the reduced minimum modulus of \( T \) corresponding to the \( m \)-seminorm introduced by A. Lebow and M. Schechter in [4]. \( \delta_1(T : I) \) was studied in [5] and there it was shown that for a Fredholm operator \( T \), \( \lim \delta_1(T^m : I)^{1/m} \) exists and is equal to the distance \( d(0, C \setminus \Phi(T : I)) \) of 0 to the complement of the Fredholm set of \( T \).

THEOREM. Let \( T \) be a Fredholm operator. Then

\[
\lim_{m \to \infty} \delta_m(T : S)^{1/m}
\]

exists and is equal to \( d(0, C \setminus \Phi(T : S)) \), the Fredholm radius of \( T \) and \( S \).

This result is closely related to the stability radius of a bundle of operators.

Received by the editors June 29, 1977.


Key words and phrases. Fredholm operators, perturbation theory.

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studied by H. Bart and D. C. Lay [1] in general and earlier by K.-H. Förster
and M. A. Kaashoek [2] in the case $X = Y$ and $S = I$. In addition, the proof
of the theorem requires a modification of the decomposition theorem of T.
Kato [3, Theorem 4]. Both stability radius and decomposition use the fol-
lowing notation (see [3], [1]).

Define subspaces $N_m = N_m(T : S)$ and $R_m = R_m(T : S)$ of $X$ by

$$N_0 = \{0\}, \quad N_m = T^{-1}SN_{m-1},$$
$$R_0 = X, \quad R_m = S^{-1}TR_{m-1}, \quad m = 1, 2, \ldots$$

The smallest number $m$ for which the null space $N(T) = N_1$ of $T$ is not
contained in $R_m$ will be denoted by $\nu(T : S)$. Further let $\Delta(T : S)$ denote
the open set of all $\lambda$ in $\Phi(T : S)$ such that $\nu(T - \lambda S : S) = \infty$.

1.1. Stability radius [1]. For $m > 1$ let $\gamma_m = \gamma_m(T : S)$ denote the supre-
num of all $c > 0$ with the property that

$$\|Tx_m\| > c \cdot d(x_m, N_m)$$

for every chain $(x_1, \ldots, x_m)$. If $T$ is a Fredholm operator, the limit
lim $\gamma_m(T : S)^{1/m}$ exists and is equal to $d(0, C \setminus [\Delta(T : S) \cup \{0\}])$, i.e. the
supremum of all $r > 0$ such that $n(T - \lambda S) = \dim N(T - \lambda S)$ and $d(T -
\lambda S) = \text{codim } R(T - \lambda S)$ are constant on $0 < |\lambda| < r$.

1.2. Suppose $n(T)$ is finite. Then for $m > 1$ there is a compact projection
$P_m$ of $X$ onto $N_m$ such that $\|P_m\| < m \cdot n(T)$. Then

$$\|(I - P_m)x_m\| \leq \|(I - P_m)d(x_m, N_m)\| \leq \|(I - P_m)\gamma_m(T : S)^{-1}\|Tx_m\|$$

for every chain $(x_1, \ldots, x_m)$ where, as usual, $0^{-1} = \infty$. Therefore, $\gamma_m$ and $\delta_m$
are related by $\gamma_m < (1 + mn(T))\delta_m$.

2.1. Decomposition [3, Theorem 4]. Let $T$ be a Fredholm operator such that
$\nu(T : S)$ is finite. There exist topological decompositions $X = X_0 \oplus X_1$ and
$Y = Y_1 \oplus Y_1$ which completely reduce $T$ and $S$. For $i = 0, 1$, let $T_i$ and $S_i$
denote the restrictions of $T$ and $S$ to $X_i$ viewed as operators into $Y_i$. Then
$\nu(T_0 : S_0) = \infty$, $S_1$ is bijective, $S_1^{-1}T_1$ is nilpotent, and $\dim X_1 = \dim Y_1 <
\infty$. As a consequence, we have $\Delta(T_0 : S_0) = \Delta(T : S) \cup \{0\}$.

2.2. Now take $0 < \rho < d(0, C \setminus (T : S))$ and let $\Delta_\rho$ denote the set of all
complex numbers $\lambda$ such that $|\lambda| < \rho$ and $\nu(T - \lambda S : S) < \infty$. By induction
we remove the finite set $\Delta_\rho$ from $\Delta(T : S)$ and obtain decompositions $X = X_\rho
\oplus X_1$ and $Y = Y_\rho \oplus Y_1$ such that $\Delta(T_\rho : S_\rho) = \Delta(T : S) \cup \Delta_\rho$, where $T_\rho$
and $S_\rho$ are restrictions of $T$ and $S$ to $X_\rho$ as in 2.1.

2.1 and 2.2 remain true if Fredholm operators are replaced by semi-Fred-
holm operators. However, this is not possible in the case 1.1, see [1, 4.1].

Now we are able to prove the theorem. It will be shown
(a) $d(0, C \setminus \Phi(T : S)) \leq \lim \inf \delta_m(T : S)^{1/m}$ and
(b) $\lim \sup \delta_m(T : S)^{1/m} < d(0, C \setminus \Phi(T : S))$.

Both parts together establish the theorem.

(a) Since $T$ is Fredholm, $d(0, C \setminus \Phi(T : S))$ is positive. Take $0 < \rho <
d(0, C \setminus \Phi(T : S))$. 1.1 and 2.2 imply that $\rho < \lim \gamma_m(T_\rho : S_\rho)^{1/m}$. For $m =$
1, 2, ..., let $P_m$ be a projection of $X$ onto $N_m(T_p : S_p)$ with $\|P_m\| < mn(T_p)$ and let $(x_1, \ldots, x_m)$ be a chain for $T$ and $S$. Furthermore let $P$ and $Q$ be the bounded projections of $X$ onto $X_1$ along $X_p$, and of $Y$ onto $Y_1$ along $Y_p$, respectively. Then $P_mP = 0$, $P_m + P \in K(X)$, and it is easy to verify that 
\[((I - P)x_1, \ldots, (I - P)x_m)\] 
is a chain for $T_p$ and $S_p$. But then
\[
\|[(I - (P_m + P)]x_m\| = \|(I - P_m)(I - P)x_m\|
\]
\[
< \|I - P_m\| \gamma_m(T_p : S_p)^{-1} \|T_p(I - P)x_1\|
\]
\[
< \|I - P_m\| \gamma_m(T_p : S_p)^{-1} \|I - Q\| \|Tx_1\|,
\]
as in 1.2. Hence
\[
\left[(1 + mn(T_p))\|I - Q\| \right]^{-1} \gamma_m(T_p : S_p) < \delta_m(T : S),
\]
and consequently
\[
\rho < \lim \gamma_m(T_p : S_p)^{1/m} < \liminf \delta_m(T : S)^{1/m},
\]
which proves (a).

(b) Take $0 < |\lambda| < \alpha < \limsup \delta_m(T : S)^{1/m}$. First, suppose $\nu(T : S) = \infty$. This restriction will be removed later with the aid of 2.1. It will be shown that $T - \lambda S$ is a Fredholm operator. There exists some $m$ and a compact operator $C = C_{a,m}$ on $X$ such that
\[
\|(I - C)x_m\| < \alpha^{-m}\|Tx_1\|,
\]
for every chain $(x_1, \ldots, x_m)$ for $T$ and $S$. Since $T$ is Fredholm and $\nu(T : S) = \infty$, that is, $N(T) \subseteq R_{m-1}$, there exists a relative inverse $L_m$ of $T$ such that $L_mTR_n \subseteq R_n$ for $n = 0, 1, \ldots, m - 1$. Since $TR_{m-1}$ has finite deficiency in $Y$, there is a projection $Q$ of $Y$ onto $TR_{m-1}$ such that $\|Q\| < 1 + m\delta(T)$. Take $y \in Y$ and put
\[
Q = (L_mS)^{-1}L_mQy, \quad i = 1, \ldots, m.
\]
$(x_1, \ldots, x_m)$ turns out to be a chain with $Tx_1 = Qy$. Consequently
\[
\|(I - C)(L_mS)^{-1}L_mQ\| < \alpha^{-m}\|Q\|,
\]
and taking $Q = I - P$, $P \in K(Y)$, we have
\[
\|(I - C)(L_mS)^{-1}L_m(I - P)S\| = \|(L_mS)^{-1} - K_m\| < \alpha^{-m}\|Q\| \|S\|
\]
with some $K_m \in K(Y)$.

Now let $\pi$ denote the canonical mapping from $B(X)$ onto $B(X)/K(X)$. Here $B(X)$ is the space of all bounded linear operators on $X$. Take any relative inverse $L$ of $T$. Then $L_m - L$ is degenerate, hence $\pi(L_mS) = \pi(LS)$, and the last inequality reads
\[
\|\pi(LS)\| < \alpha^{-m}(1 + m\delta(T))\|S\|.
\]
This implies $r_\alpha < \alpha^{-1} < |\lambda|^{-1}$.

Here $r_\alpha$ is the spectral radius of $\pi(LS)$. But then $\lambda^{-1}\pi(I) - \pi(LS)$ is invertible in $B(X)/K(X)$, hence $I - \lambda LS$ is Fredholm and so is $T - \lambda TLS$. 
Since \( L \) is a relative inverse of \( T \), \( TL = I - R \), where \( R \) is a compact projection. So \( T - \lambda S \) is a Fredholm operator if \( \nu(T : S) = \infty \). Now suppose \( \nu(T : S) \) is finite. Then \( \nu(T_0 : S_0) = \infty \) by 2.1. Let \( P_0 \) be the bounded projection of \( X \) onto \( X_0 \) along \( X_1 \). Starting with a chain for \( T_0 \) and \( S_0 \) we obtain
\[
0 < |\lambda| < \lim_{m} \sup \delta_m(T : S)^{1/m} < \lim_{m} \sup \delta_m(T_0 : S_0)^{1/m},
\]
and by the preceding argument \( T_0 - \lambda S_0 \) is Fredholm. Since \( S_1^{-1}T_1 \) is nilpotent and \( \lambda \neq 0 \), \( T_1 - \lambda S_1 \) is bijective, thus \( T - \lambda S \) is Fredholm. This proves (b).

**Corollary 1.** Let \( T \) be a Fredholm operator. Then \( \Phi(T : S) = C \) if and only if \( \lim_{m} \delta_m^{1/m} = \infty \), i.e. if and only if for each \( \varepsilon > 0 \) and sufficiently large \( m \) there are compact operators \( C_{t,m} \) on \( X \), such that for every chain \( (x_1, \ldots, x_m) \)
\[
\|x_m\| < \varepsilon^m \|T x_1\| + \|C_{t,m} x_m\|,
\]

Let \( \hat{X} \) be \( D(T) \) endowed with the graph norm \( \|x\|_T = \|x\| + \|T x\| \), let \( \hat{T} \) and \( \hat{S} \) be the operators \( T \) and \( S \) considered as maps from \( \hat{X} \) into \( Y \), and let \( i_T \) be the inclusion map of \( \hat{X} \) into \( X \). Then \( \hat{X} \) is a Banach space, \( \hat{T}, \hat{S}, i_T \) are bounded, and \( \hat{T} = Ti_T, \hat{S} = Si_T \). Put
\[
\delta_m = \delta_m(T : S) = \sup_{K \in K(\hat{X}, X)} \inf_{(x_1, \ldots, x_m)} \frac{\|\hat{T} x_1\|}{\|(i_T - K x_m)\|},
\]
where \( (x_1, \ldots, x_m) \) is a chain for \( \hat{T} \) and \( \hat{S} \). Since \( \hat{T} x_1 = T x_1 \) and \( C \in K(X) \) implies \( C i_T \in K(\hat{X}, X) \), we have \( \delta_m(T : S) \leq \delta_m(T : S) \).

**Corollary 2.** Let \( T \) be a Fredholm operator. Then \( \lim_{m} \delta_m^{1/m} = d(0, C \setminus \Phi(T : S)) \) and, as a consequence \( \Phi(T : S) = C \), if \( i_T \) is compact.

**Proof.** By the preceding remark we have \( d(0, C \setminus \Phi(T : S)) \leq \lim inf \delta_m^{1/m} \). Replacing \( T \) by \( \hat{T} \), \( S \) by \( \hat{S} \), and \( B(X) / K(X) \) by \( B(\hat{X}, X) / K(\hat{X}, X) \) in part (b) of the theorem, we obtain \( \lim sup \delta_m^{1/m} \leq d(0, C \setminus \Phi(T : S)) \). If \( i_T \) is compact, then \( \delta_m = \infty \), hence the corollary.

**Remark [5].** Let \( X = Y \) be a complex Hilbert space, suppose \( S = I \), and let \( T \) be a densely defined normal Fredholm operator. Then \( d(0, C \setminus \Phi(T : I)) = \delta_1 = \delta_1 \). If moreover \( d(0, C \setminus \Phi(T : I)) < \infty \), i.e. the Fredholm set of \( T \) is not the whole plane, then there exists a compact operator \( K \) on \( X \) such that \( \delta_1(T : I) = \gamma(T - K) \), where \( \gamma(T - K) \) denotes the reduced minimum modulus of \( T - K \). These facts use the resolution of the identity corresponding to \( T \).

**References**


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