REPRESENTABLE MONOIDS

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Abstract. A representable monoid is one with enough representative functions to separate points. It is shown that the monoid algebra of a representable monoid is a proper algebra. In particular, the group algebra of a residually-finite group is a proper algebra. It is also shown that the free product of two representable monoids is again representable.

1. Introduction. Let $S$ be a monoid, $K$ a field. For $s$ in $S$, let $r_s$ and $l_s$ denote right and left multiplication by $s$ in $S$. A function $f: S \to K$ is called representative if the right translates $s \cdot f = f r_s$ of $f$ for $s$ in $S$ span a finite-dimensional $K$-vector space. Let $R(S)$ denote the set of representative functions on $S$. $R(S)$ is a bialgebra (Hopf algebra, possibly without an antipode). $R$ is a contravariant functor from monoids to bialgebras. We called $S$ representable in [T I] if $R(S)$ separates points in $S$. In this note, we study the class of representable monoids. See [H] for a discussion of representative functions. There $S$ is a group, but the basic ideas carry over for $S$ a monoid.

There is a close relationship between representable monoids and proper algebras. Recall that an algebra $A$ is called proper if there are enough linear functions representative with respect to the multiplicative monoid structure of $A$ to separate points of $A$. See [S], [T I] and [T II] for alternate characterizations of proper algebras. We show here that $S$ is representable if and only if the monoid algebra $K[S]$ is proper. We use this result to show that the coproduct of two representable monoids is representable.

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2. The class of representable monoids. $K[S]$ will denote the monoid algebra of $S$ over $K$.

Lemma 1. $S$ is representable if and only if for any $s \neq t$ in $S$, there is a representation $\rho$ of $S$ on a finite-dimensional $K$-vector space for which $\rho(s) \neq \rho(t)$.

Proof. Let $S$ be representable, and $s \neq t$ in $S$. Choose $f$ in $R(S)$ with $f(s) \neq f(t)$. Then $K[S] \cdot f$, the span of the right translates of $f$ by elements of
$S$ is a (left)$K[S]$-module which is finite-dimensional over $K$. $(s \cdot f)(1) = f(s)$ and $(t \cdot f)(1) = f(t)$, so $s \cdot f \neq t \cdot f$ and $\rho(s) \neq \rho(t)$, where $\rho$ is the representation of $S$ on $K[S] \cdot f$. Conversely, for $s \neq t$ in $S$, let $\rho$ be a representation of $S$ on a finite-dimensional space $V$ so that $\rho(s) \neq \rho(t)$. If $p_y$ is a coordinate function on $\text{Hom}_K(V, V)$ separating $\rho(s)$ and $\rho(t)$, then $p_y \rho$ is a representative function on $S$ separating $s$ and $t$. This follows from the formula for matrix multiplication. For if $a, b$ are in $S$, then

$$(p_y \rho_{p_a})(b) = p_y \rho_{p_a}(ba) = p_y(\rho(b)\rho(a))$$

$$= \sum_k (p_{ik} \rho(b))(p_{ik} \rho(a)).$$

Thus $p_y \rho_{p_a} = \sum_k (p_{ik} \rho(a))(p_{ik} \rho)$ lies in the span of the $p_{ik} \rho$.

We included Lemma 1 for completeness. The idea is that $R(S)$ consists of all coordinate functions arising from finite-dimensional representations of $S$. See [H, pp. 14-15], for a discussion of this.

**Proposition 2.** A submonoid of a representable monoid is representable.

**Proof.** This is clear by restriction of finite-dimensional representations.

**Proposition 3.** Let $\{S_i| i \in I\}$ be a collection of monoids. Then $S = \prod_i S_i$ is representable if and only if each $S_i$ is representable.

**Proof.** If $S$ is representable, then each $S_i$ is representable by Proposition 2. Conversely let each $S_i$ be representable. For $s \neq t$ in $S$, let $p_j$ be a projection of $S$ onto $S_i$ for which $p_j(s) \neq p_j(t)$. Let $f \in R(S_j)$, $f(p_j(s)) \neq f(p_j(t))$. Then $fp_j$ is in $R(S)$ and separates $s$ and $t$.

**Corollary 4.** A direct sum (weak direct product) of monoids $\{S_i| i \in I\}$ is representable if and only if each $S_i$ is representable.

**Proof.** This follows from Propositions 2 and 3.

**Lemma 5.** Let $A$ be a $K$-algebra which is an integral domain. Let $\{a_i| 1 \leq i \leq n\}$ be distinct elements of $A$. Then the elements $\{(1, a_i, a_i^2, \ldots, a_i^{n-1})| 1 \leq i \leq n\}$ of $A^n$ are linearly independent over $K$.

**Proof.** By the Vandermonde determinant, $\{(1, a_i, a_i^2, \ldots, a_i^{n-1})| 1 \leq i \leq n\}$ are linearly independent over $A$, hence also over $K$.

**Lemma 6.** Let $V$ be a vector space over $K$. For $i \geq 0$, let $V^{(i)}$ denote $V \otimes V \otimes \cdots \otimes V$ ($i$ factors). If $v_1, \ldots, v_n$ are distinct elements of $V$, then the elements $\{(1, v_i, v_i \otimes v_j, \ldots, v_i \otimes \cdots \otimes v_i)| 1 \leq i < n\}$ of $K \oplus V \oplus V^{(2)} \oplus \cdots \oplus V^{n-1}$ are linearly independent over $K$.

**Proof.** Let $S^{(i)}$ denote the $i$th symmetric tensor product of $V$. Then the elements $\{(1, v_i, v_i^2, \ldots, v_i^{n-1})| 1 \leq i < n\}$ of the symmetric algebra $S(V)$ are linearly independent over $K$ by Lemma 5. Since $S(V)$ is a homomorphic image of the tensor algebra $T(V)$, the elements $\{(1, v_i, v_i \otimes v_j, \ldots, v_i \otimes \cdots \otimes v_i)| 1 \leq i < n\}$ of $T(V)$ are also linearly independent over $K$. 

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Lemma 7. Let \( W \) be a finite-dimensional vector space over \( K \). Let \( \{T_i|1 < i < n\} \) be distinct elements of \( V = \text{Hom}_K(W, W) \). For \( 1 < i < n \), let \( \tilde{T}_i \) denote the diagonal extension of \( T_i \) to \( \text{Hom}^{\oplus}_{K}(X, X) \), where \( X = K \oplus W \oplus W^{(2)} \oplus \cdots \oplus W^{(n-1)} \subset T(W) \). Then \( \{\tilde{T}_i|1 < i < n\} \) are linearly independent over \( K \).

Proof. By Lemma 6, \((\{1, T_1, T_1 \otimes T_1, \ldots, T_1 \otimes \cdots \otimes T_1\}|1 < i < n)\) of \( K \otimes V \oplus V(2) \oplus \cdots \oplus V^{(n-1)} \) are linearly independent over \( K \). Identifying \( (\text{hom}_K(W, W))^{(0)} \) with \( \text{Hom}_K(W^{(0)}, W^{(0)}) \) for \( 0 < j < n - 1 \), the element \((1, T_i, T_i \otimes T_i, \ldots, T_i \otimes \cdots \otimes T_i)\) corresponds to \( \tilde{T}_i \). Thus \( \{\tilde{T}_i|1 < i < n\} \) are linearly independent over \( K \).

Theorem 8. A monoid \( S \) is representable if and only if the monoid algebra \( K[S] \) is proper.

Proof. If \( K[S] \) is proper, then \( S \) is representable by Proposition 3.6 of [T I]. Conversely, let \( S \) be representable. Let \( \alpha_1s_1 + \cdots + \alpha_ns_n \) be a nonzero element of \( K[S] \), where \( s_1, \ldots, s_n \) are distinct in \( S \), and \( \alpha_1, \ldots, \alpha_n \) are nonzero elements of \( K \). For \( i \neq j \), \( 1 < i, j < n \), let \( \rho_{ij} \) be a representation of \( S \) on a finite-dimensional vector space \( V_{ij} \) for which \( \rho_{ij}(s_i) \neq \rho_{ij}(s_j) \). Set \( V = \Sigma_{i,j=1}^{n} \bigoplus V_{ij} \), and \( \rho = \Sigma_{i,j=1}^{n} \bigoplus \rho_{ij} \). Then \( V \) is finite-dimensional over \( K \) and \( \rho(s_i) \neq \rho(s_j) \) for \( i \neq j, 1 < i, j < n \). Let \( \tilde{\rho} \) be the diagonal extension of \( \rho \) to a representation of \( S \) and \( K[S] \) on the space \( W = K \oplus V \oplus V^{(2)} \oplus \cdots \oplus V^{(n-1)} \). Then \( \{\tilde{\rho}(s)|1 < i < n\} \) are linearly independent by Lemma 7. Hence \( \tilde{\rho}(\Sigma_{i=1}^{n} \alpha_is_i) = \Sigma_{i=1}^{n} \alpha_i\tilde{\rho}(s_i) \neq 0 \) in \( \text{Hom}_K(W, W) \). Hence the kernel of \( \tilde{\rho} \) is a cofinite ideal of \( K[S] \) not containing \( \Sigma_{i=1}^{n} \alpha_is_i \). Hence \( K[S] \) is proper by Lemma 6.1.0 of [S].

Corollary 9. If \( S \) is a residually finite monoid, then \( K[S] \) is a proper algebra.

Proof. We show \( S \) is representable, so that the result follows from Theorem 8. Let \( s \neq t \) in \( S \). Let \( \rho : S \to H \) be a homomorphism of \( S \) to a finite monoid \( H \) so that \( \rho(s) \neq \rho(t) \). \( \rho \) induces a representation \( \tilde{\rho} \) of \( S \) on \( K[H] \) by left translation. \( \tilde{\rho}(s)(1) = \rho(s) \neq \rho(t) = \tilde{\rho}(t)(1) \), so \( \tilde{\rho}(s) \neq \tilde{\rho}(t) \).

If \( G \) is a residually finite group, then it is residually finite as a monoid, and so \( K[G] \) is a proper algebra. This was noted by A. Rosenberg in Theorem 2 of [ROS].

Theorem 10. Let \( S \) and \( T \) be representable monoids. Then the free product \( S \ast T \) is a representable monoid.

Proof. \( K[S] \) and \( K[T] \) are proper by Theorem 8. Hence \( K[S] \ast K[T] \) (the free product of algebras) is proper by Proposition 3.4.2 of [T II]. But \( K[S] \ast K[T] = K[S \ast T] \). Hence \( S \ast T \) is representable by Theorem 8.

We conclude by noting that the class of representable monoids is not closed under homomorphic images. This was pointed out in Proposition 2.6 of [T I] if \( K \) is a finite field, but we give examples here for any field \( K \). First
note that any free monoid is representable by Proposition 2.5 of [T I]. Hence it suffices to exhibit a nonrepresentable monoid. Let $G$ be a finitely-generated infinite simple group. For examples of such, see §5.1 of [ROB]. We claim that $R(G) = K$, the constant functions on $G$, so that $G$ is not representable. Let $\rho$ be a (monoid) representation of $G$ on a finite-dimensional vector space of dimension $n$. $\rho(G)$ is a linear group of $n$ by $n$ matrices over $K$. Since $G$ is a simple group, $\rho$ is either one-to-one or trivial. If $\rho$ is one-to-one, then $\rho(G)$ is a finitely-generated infinite linear group. By Malcev's theorem (see Corollary 4.4(ii) of [W]), $\rho(G)$ is not simple. Hence $\rho$ is trivial, i.e. a one-dimensional representation. It follows that $R(G) = K$.

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