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A GENERALIZATION OF A THEOREM OF TATCHELL

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ABSTRACT. Necessary and sufficient conditions for $\Sigma a_n \varepsilon_n$ to be summable $|A, \lambda_n|$, whenever Σa_n is convergent, have been obtained. The sufficiency part of this result has also been improved.

1. Let $\Sigma_{n=0}^{\infty} a_n$ be a given infinite series and let $\{\lambda_n\}$ be a sequence of nonnegative numbers increasing to infinity. For any $k > 0$, we write

$$A^k(t) = \sum_{\lambda_n < t} (t - \lambda_n)^k a_n.$$

We say that $\Sigma_{n=0}^{\infty} a_n$ is summable (R, λ_n, k) to S if $t^{-k} A^k(t) \to S$ as $t \to \infty$. If, in addition, $t^{-k} A^k(t)$ is of bounded variation in $(0, \infty)$, we say that the series $\Sigma_{n=0}^{\infty} a_n$ is summable $|R, \lambda_n, k|$ to S. The discontinuous Riesz means (R^*, λ_n, k) are obtained by restricting t to the sequence $\{\lambda_n\}$. If $\Sigma a_n e^{-\lambda_n x}$ is convergent for all positive x and $f(x) = \Sigma a_n e^{-\lambda_n x} \to S$ when $x \to 0$, then we say that the series $\Sigma_{n=0}^{\infty} a_n$ is summable (A, λ_n) to sum S, and write $\Sigma_{n=0}^{\infty} a_n = S(A, \lambda_n)$. When $\lambda_n = n$, the (A, λ_n) method is the Abel method. The series $\Sigma_{n=0}^{\infty} a_n$ is said to be absolutely summable (A, λ_n) or summable $|A, \lambda_n|$, if the series $\Sigma_{n=0}^{\infty} a_n e^{-\lambda_n x}$ is convergent for all positive values of x and the sum function $f(x) = \Sigma_{n=0}^{\infty} a_n e^{-\lambda_n x}$ is of bounded variation in $[0, \infty)$. We also define $\Delta \varepsilon_n = \varepsilon_n - \varepsilon_{n+1}$.

2. Necessary and sufficient conditions on $\{\varepsilon_n\}$ in order that $\Sigma a_n \varepsilon_n$ should be absolutely Abel summable whenever Σa_n converges were obtained by Tatchell [3] in 1954 in the form of the following

THEOREM. *Necessary and sufficient conditions for $\Sigma a_n \varepsilon_n$ to be summable $|A|$, whenever Σa_n is convergent are*

$$\sum |\Delta \varepsilon_n| < \infty, \tag{2.1}$$

and

$$\sum |\varepsilon_n|/n < \infty. \tag{2.2}$$

The object of this paper is to generalize the above theorem by replacing absolute Abel summability $|A|$ by summability $|A, \lambda_n|$ for any sequence $\{\lambda_n\}$ satisfying weaker conditions. In what follows we prove the following theorem.

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THEOREM 1. *Necessary and sufficient conditions for $\sum a_n \varepsilon_n$ to be summable $|A, \lambda_n|$, whenever $\sum a_n$ is convergent, are*

$$\sum |\Delta\varepsilon_n| < \infty, \tag{2.3}$$

$$\sum |\varepsilon_n|(1-\mu_n) < \infty, \tag{2.4}$$

where

$$\mu_n = \lambda_n/\lambda_{n+1} \quad \textit{and} \quad \lambda_{n+1} = O(\lambda_n).$$

Taking $\lambda_n = n$ in our theorem, the above-mentioned theorem of Tatchell follows as a special case.

3. To prove the theorem we need the following lemmas:

LEMMA 1. *If $H(p) = \alpha_p$ is a transformation from a Banach space B to the space L, and if $h_x(p) = \alpha_p(x)$ is a continuous linear functional on B whenever $x > 0$, then $H(p)$ is a bounded linear operator.*

L in this context is really $\mathring{L}$ as defined in Dunford and Schwartz, *Linear operators*. I, p. 119.

The proof of this lemma may be found in **[4]**.

LEMMA 2. *If a sequence $\{\varepsilon_n\}$ has the property that the function*

$$\sum_0^\infty S_n\varepsilon_n\left\{\frac{d}{dx}\Delta e^{-\lambda_n x}\right\}$$

is defined and has a finite Lebesgue integral on $[0, \infty)$, whenever $\{S_n\}$ is a convergent sequence, then there is a number H such that

$$\int_0^\infty \left|\sum_{n=0}^\infty S_n\varepsilon_n\left\{\frac{d}{dx}\Delta e^{-\lambda_n x}\right\}\right| dx < H\,\overline{\mathrm{bd}}\,|S_n|$$

for every convergent sequence $\{S_n\}$.

PROOF. If

$$h_x(S) = \sum_{n=0}^\infty S_n\varepsilon_n\left\{\frac{d}{dx}\Delta e^{-\lambda_n x}\right\}$$

is defined whenever $S = \{S_n\}$ is a convergent sequence, then h_x is a linear functional on the Banach space C. Therefore, by hypothesis, h_x is a linear functional on C whenever $0 < x < \infty$.

Also by hypothesis

$$\sum_{n=0}^\infty S_n\varepsilon_n\left\{\frac{d}{dx}\Delta e^{-\lambda_n x}\right\}$$

is in the space L whenever S is in C. Hence the lemma follows from Lemma 1.

LEMMA 3. *If a sequence $\{p_n\}$ of elements in a Banach space B has the property that there is a number H such that for every nonnegative integer K and* *every set of real numbers* $\theta_0, \theta_1, \theta_2, \ldots, \theta_k$,

$$\left\| \sum_{n=0}^{k} e^{i\theta_n} p_n \right\| \leqslant H,$$

then $\sum_{n=0}^{\infty} |f(p_n)| < \infty$, *for every linear functional* f *on* B.

PROOF. Let f be a linear functional on B, and let $\theta_n = -\arg f(p_n)$.

$$\sum_{n=0}^{k} |f(p_n)| = \sum_{n=0}^{k} e^{i\theta_n} f(p_n) \leqslant H\|f\|,$$

whence the result follows.

LEMMA 4. *The necessary and sufficient conditions that* $\gamma(a) = \sum_{k=0}^{\infty} g_k(a) c_k$ *should tend to a finite limit as* $a \to \infty$ *whenever* $\sum c_k$ *is convergent are*:

(i) $\sum_{k=0}^{\infty} |g_k(a) - g_{k+1}(a)| \leqslant M$ *independently of* $a > a'$;

(ii) $\lim_{a \to \infty} g_k(a) = \beta_k$ *for every fixed* k.

The proof of this lemma may be found in **[5]**.

LEMMA 5. *A necessary and sufficient condition for* $\sum a_n \varepsilon_n$ *to be summable* (A, λ_n) *whenever* $\sum a_n$ *is convergent is that* $\sum |\Delta \varepsilon_n| < \infty$.

It may be remarked that if we take $\lambda_n = n$ we get the following result of Bosanquet [Proc. London Math. Soc. (2) **50** (1948), Lemma 9].

LEMMA A. *A necessary and sufficient condition for* $\sum a_n \varepsilon_n$ *to be summable* (A) *whenever* $\sum a_n$ *is convergent is that* $\sum |\Delta \varepsilon_n| < \infty$.

PROOF OF LEMMA 5. Since $\sum a_n \varepsilon_n$ is summable (A, λ_n) we have

$$\sum_{n=1}^{\infty} a_n \varepsilon_n e^{-\lambda_n/t} \quad \text{is convergent for } t > 0$$

and

$$\lim_{t \to \infty} \sum_{n=1}^{\infty} a_n \varepsilon_n e^{-\lambda_n/t} \quad \text{exists.}$$

Applying Lemma 4 we have

$$\sum_{n=1}^{\infty} \left|\Delta\left(\varepsilon_n e^{-\lambda_n/t}\right)\right| \leqslant M \quad \text{independently of } t > t_0.$$

This implies that $\sum_{n=1}^{\infty} |\Delta \varepsilon_n| < \infty$.

The proof of the sufficiency part of the lemma is immediate for $\sum a_n$ is convergent and $\{\varepsilon_n\}$ is a sequence of bounded variation.

LEMMA 6. *We have*

$$\int_0^{\infty} |d\Delta e^{-\lambda_n x}| = 2(1 - \mu_n)(\mu_n)^{\mu_n/(1-\mu_n)}.$$

The proof is simple.

LEMMA 7. *If* $\mu_n = \lambda_n/\lambda_{n+1}$, *then the following conditions are equivalent.*
(i) $\Sigma|\varepsilon_n|(1 - \mu_n) < \infty$,
(ii) $\Sigma|\varepsilon_n|(1 - \mu_n)\,\mu_n^{\mu_n/(1-\mu_n)} < \infty$.

PROOF. Since $0 \leqslant \mu_n < 1$, so $e^{-1} < \mu_n^{\mu_n/(1-\mu_n)} < 1$. Hence (i) and (ii) are equivalent.

LEMMA 8. *Let M, a be constants with $M > 1$, $a > 0$ and*
(i) $a \log M < 2\pi$.
Then for $1/M \leqslant u < 1$ we have $|1 - u^{-ai}|/(1 - u) \geqslant C$, where $C > 0$ is a constant (depending on a and M).

PROOF. Given (i), the function $(1 - u^{-ai})(1 - u)^{-1}$ is continuous and nonzero on $1/M \leqslant u < 1$, and has a nonzero limit as $u \to 1$. This proves the lemma.

LEMMA 9. *For $k = 1$, $|R, \lambda_n, k| \sim |R^*, \lambda_n, k|$.*

PROOF. Let

$$\gamma(t) = t^{-k} \sum_{\lambda_n < t} (t - \lambda_n)^k a_n.$$

$|R, \lambda_n, k|$ means that

$$\int_0^\infty |d\gamma(t)| < \infty. \tag{i}$$

$|R^*, \lambda_n, k|$ means

$$\sum_{n=1}^{\infty} |\gamma(\lambda_n) - \gamma(\lambda_{n+1})| < \infty. \tag{ii}$$

Now if $k = 1$, then for $\lambda_n \leqslant t \leqslant \lambda_{n+1}$,

$$\gamma(t) = \frac{1}{t} \sum_{\nu=0}^{n} (t - \lambda\nu)a_\nu = A - B/t,$$

where $A = \Sigma_{\nu=0}^{n} a_\nu$, $B = \Sigma_{\nu=0}^{n} \lambda_\nu a_\nu$. Thus, for a given n, A, B are constants (though their values will vary with n).

Hence, for a given n, $\gamma(t)$ is monotonic in each interval $[\lambda_n, \lambda_{n+1}]$ (increasing if $B > 0$ and decreasing if $B < 0$). Hence

$$\int_{\lambda_n}^{\lambda_{n+1}} |d\gamma(t)| = |\gamma(\lambda_{n+1}) - \gamma(\lambda_n)|$$

so that (i) and (ii) are equivalent.

NOTE. It is known [6] that $|R, \lambda_n, k| \sim |R^*, \lambda_n, k|$ for $0 < k < 1$.

LEMMA 10. *For any $k > 0$, $|R, \lambda_n, k| \Rightarrow |A, \lambda_n|$.*

The proof of this lemma may be found in [7].

4. Proof of the theorem.

Sufficiency. Since Σa_n is convergent, by virtue of (2.3) and Abel's test we see that $\Sigma a_n \varepsilon_n e^{-\lambda_n x}$ is convergent for all positive x, so that writing

$$\alpha(x) = \sum_{n=0}^{\infty} a_n \varepsilon_n e^{-\lambda_n x},$$

we have

$$\begin{aligned}\alpha(x) &= \sum_{n=0}^{\infty} S_n \Delta(\varepsilon_n e^{-\lambda_n x}), \qquad S_n = a_0 + a_1 + \cdots + a_n \\ &= \sum_{n=0}^{\infty} S_n e^{-\lambda_{n+1} x} \Delta\varepsilon_n + \sum_{n=0}^{\infty} S_n \varepsilon_n \Delta e^{-\lambda_n x} \\ &= \alpha_1(x) + \alpha_2(x) \quad \text{(say)}. \end{aligned} \tag{4.1}$$

In order to prove that $\Sigma a_n \varepsilon_n$ is summable $|A, \lambda_n|$ it is sufficient to show that $\alpha_1(x)$ and $\alpha_2(x)$ are functions of bounded variation on $[0, \infty)$, that is to say,

$$\int_0^\infty |d\alpha_1(x)| < \infty, \tag{4.2}$$

and

$$\int_0^\infty |d\alpha_2(x)| < \infty. \tag{4.3}$$

Now

$$\begin{aligned}\int_0^\infty |d\alpha_1(x)| &= \int_0^\infty \left| d \sum_{n=0}^{\infty} S_n e^{-\lambda_{n+1} x} \Delta\varepsilon_n \right| \\ &\leqslant \sum_{n=0}^{\infty} |S_n| \cdot |\Delta\varepsilon_n| \int_0^\infty |de^{-\lambda_{n+1} x}| \\ &\leqslant \overline{\mathrm{bd}}\, |S_n| \sum_{n=0}^{\infty} |\Delta\varepsilon_n| < \infty. \end{aligned}$$

Also by virtue of Lemmas 6 and 7 and condition (2.4) we observe that

$$\begin{aligned}\int_0^\infty |d\alpha_2(x)| &= \int_0^\infty \left| d \sum_{n=0}^{\infty} S_n \varepsilon_n \Delta e^{-\lambda_n x} \right| \\ &\leqslant \sum_{n=0}^{\infty} |S_n| \cdot |\varepsilon_n| \int_0^\infty |d\Delta e^{-\lambda_n x}| \\ &\leqslant 2\,\overline{\mathrm{bd}}\, |S_n| \sum_{n=0}^{\infty} |\varepsilon_n| (1 - \mu_n)\, \mu_n^{\mu_n/(1-\mu_n)} \\ &= 2\,\overline{\mathrm{bd}}\, |S_n| \sum_{n=0}^{\infty} |\varepsilon_n| (1 - \mu_n) < \infty. \end{aligned}$$

This completes the sufficiency part of the theorem.

Necessity. Since summability $|A, \lambda_n|$ implies summability (A, λ_n), it follows from Lemma 5 that (2.3) is necessary. Also from (4.1), since (2.3) holds, we have, as before, $\int_0^\infty |d\alpha_1(x)| < \infty$.

Since by hypothesis $\alpha(x)$ is of bounded variation it follows that $\alpha_2(x)$ is also of bounded variation. Therefore

$$\int_0^\infty \left| \sum_{n=0}^\infty S_n \varepsilon_n \left[\frac{d}{dx} \Delta e^{-\lambda_n x} \right] \right| dx = \int_0^\infty \left| \frac{d}{dx} \sum_{n=0}^\infty S_n \varepsilon_n \Delta e^{-\lambda_n x} \right| dx$$

$$= \int_0^\infty \left| d \sum_{n=0}^\infty S_n \varepsilon_n e^{-\lambda_n x} \right| < \infty \tag{4.4}$$

for every convergent sequence $\{S_n\}$. Applying Lemma 2 we have

$$\int_0^\infty \left| \sum_{n=0}^\infty S_n \varepsilon_n \left[\frac{d}{dx} \Delta e^{-\lambda_n x} \right] \right| dx \leqslant H \,\overline{\mathrm{bd}}\, |S_n|$$

for every sequence $\{S_n\}$. In particular, we have

$$\int_0^\infty \left| \sum_{n=0}^k e^{i\theta_n} \varepsilon_n \left[\frac{d}{dx} \Delta e^{-\lambda_n x} \right] \right| dx \leqslant H \tag{4.5}$$

for every nonnegative integer k and every set of real numbers $\theta_0, \theta_1, \ldots, \theta_k$. Again the sequence $\{\varepsilon_n (d/dx) \Delta e^{-\lambda_n x}\} \in L$, and so

$$\left\| \sum_{n=0}^k e^{i\theta_n} \varepsilon_n \frac{d}{dx} \Delta e^{-\lambda_n x} \right\| \leqslant H,$$

by virtue of inequality (4.5). Hence, by Lemma 3 we have

$$\sum_{n=0}^\infty \left| f\left(\varepsilon_n \frac{d}{dx} \Delta e^{-\lambda_n x} \right) \right| < \infty$$

for every linear functional f on L.

But, for a given bounded measurable complex function $\phi(x)$,

$$f_\phi(\psi) = \int_0^\infty \phi(x) \psi(x)\, dx$$

is a linear functional on L. Therefore

$$\sum_{n=0}^\infty \left| \int_0^\infty \phi(x) \varepsilon_n \frac{d}{dx} \Delta e^{-\lambda_n x}\, dx \right| < \infty.$$

This implies that

$$\sum_{n=0}^\infty |\varepsilon_n| \left| \int_0^\infty \phi(x) \frac{d}{dx} \Delta e^{-\lambda_n x}\, dx \right| < \infty. \tag{4.6}$$

Since $\lambda_{n+1} = O(\lambda_n)$ so there exists some constant M such that, for all $n \geqslant 1$, $\lambda_{n+1} \leqslant M\lambda_n$. Given this M, choose $a > 0$ so that $a \log M < 2\pi$. Then apply (4.6) with $\phi(x) = x^{ai}$. (This choice is suggested by the argument of [3].) Now

$$\int_0^\infty \phi(x) d(\Delta e^{-\lambda_n x}) = \Gamma(1 + ia)(-\lambda_n^{-ai} + \lambda_{n+1}^{-ai})$$

$$= \Gamma(1 + ia) \lambda_{n+1}^{-ai} (1 - \mu_n^{-ai}).$$

But, for sufficiently large n, $\mu_n \geqslant 1/M$. Hence by Lemma 8,

$$\left| \int_0^\infty \phi(x) d(\Delta e^{-\lambda_n x}) \right| \geqslant C |\Gamma(1 + ia)| (1 - \mu_n).$$

It therefore follows from (4.6) that $\Sigma_{n=0}^{\infty}|\varepsilon_n|(1-\mu_n) < \infty$ as required.

5. The sufficiency part of the theorem can be improved. If (2.3) and (2.4) hold, then, in fact, $\Sigma_{n=0}^{\infty} a_n\varepsilon_n$ is summable $|R, \lambda_n, 1|$ whenever $\Sigma_{n=0}^{\infty} a_n$ is convergent. By Lemma 10 this is stronger than the present result. For summability $|R, \lambda_n, 1|$ implies summability $|A, \lambda_n|$ when the (A, λ_n) method is applicable, i.e., whenever

$$\sum_{n=0}^{\infty} a_n\varepsilon_n e^{-\lambda_n x} \tag{5.1}$$

converges for all $x > 0$. But (2.3) alone is enough to ensure that whenever $\Sigma_{n=0}^{\infty} a_n$ converges, $\Sigma_{n=0}^{\infty} a_n\varepsilon_n$ converges, and thus (5.1) certainly converges.

Hence our Theorem 1 can be improved in the following way.

THEOREM 2. *Conditions* (2.3) *and* (2.4) *are sufficient for* $\Sigma_{n=0}^{\infty} a_n\varepsilon_n$ *to be summable* $|R, \lambda_n, 1|$ *whenever* $\Sigma_{n=0}^{\infty} a_n$ *converges, and necessary for it to be summable* $|A, \lambda_n|$ *whenever* $\Sigma_{n=0}^{\infty} a_n$ *converges and* $\lambda_{n+1} = O(\lambda_n)$.

PROOF. In the light of Theorem 1, it is sufficient to show that $\Sigma_{n=0}^{\infty} a_n\varepsilon_n$ is summable $|R, \lambda_n, 1|$ whenever $\Sigma_{n=0}^{\infty} a_n$ converges. To prove our assertion it is enough, by Lemma 9, to consider the discontinuous Riesz means, so write

$$t_n = -\frac{1}{\lambda_{n+1}} \sum_{k=0}^{n} (\lambda_{n+1} - \lambda_k) a_k \varepsilon_k.$$

Suppose that $\Sigma_{k=0}^{\infty} a_k$ converges and that (2.3) and (2.4) hold. Suppose without loss of generality that $a_0 = 0$. Then, for $n \geqslant 1$,

$$\begin{aligned} t_n - t_{n-1} &= (1/\lambda_n - 1/\lambda_{n+1}) \sum_{k=1}^{n} \lambda_k a_k \varepsilon_k \\ &= (1/\lambda_n - 1/\lambda_{n+1}) \left[\sum_{k=1}^{n-1} S_k \lambda_{k+1} \Delta\varepsilon_k + \sum_{k=1}^{n-1} S_k \varepsilon_k \Delta\lambda_k + S_n \varepsilon_n \lambda_n \right] \\ &= b_n + c_n + d_n \quad \text{(say)}. \end{aligned}$$

Supposing that for all n, $|S_n| \leqslant K$, we have

$$\begin{aligned} \sum_{n=1}^{\infty} |b_n| &\leqslant K \sum_{k=1}^{\infty} |\Delta\varepsilon_k| \lambda_{k+1} \sum_{n=k+1}^{\infty} (1/\lambda_n - 1/\lambda_{n+1}) \\ &= K \sum_{k=1}^{\infty} |\Delta\varepsilon_k| < \infty \quad \text{by (2.3)}. \end{aligned}$$

$$\begin{aligned} \sum_{n=1}^{\infty} |c_n| &\leqslant K \sum_{k=1}^{\infty} |\varepsilon_k|\, |\Delta\lambda_k| \sum_{n=k+1}^{\infty} (1/\lambda_n - 1/\lambda_{n+1}) \\ &= K \sum_{k=1}^{\infty} |\varepsilon_k| (\lambda_{k+1} - \lambda_k)/\lambda_{k+1} \\ &= K \sum_{k=1}^{\infty} |\varepsilon_k| (1 - \lambda_k/\lambda_{k+1}) = K \sum_{k=1}^{\infty} |\varepsilon_k| (1 - \mu_k) < \infty \quad \text{by (2.4)}. \end{aligned}$$

$$\sum_{n=1}^{\infty} |d_n| < K \sum_{n=1}^{\infty} |\varepsilon_n| \lambda_n (1/\lambda_n - 1/\lambda_{n+1})$$

$$= K \sum_{n=1}^{\infty} |\varepsilon_n|(1 - \lambda_n/\lambda_{n+1}) = K \sum_{n=1}^{\infty} |\varepsilon_n|(1 - \mu_n)$$

$$< \infty \quad \text{by (2.4).}$$

Hence the result.

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REFERENCES

1. G. H. Hardy, *Divergent series*, Oxford Univ. Press, London, 1949.

2. M. W. Orlicz, *Beitrage zur Theorie der Orthogonalentwicklungen*. II, Studia Math. **1** (1929), 241–255.

3. J. B. Tatchell, *A note on a theorem by Bosanquet*, J. London Math. Soc. **29** (1954), 207–211.

4. ______, *On some integral transforms*, Proc. London Math. Soc. 3 (1953), 257–267.

5. P. Dienes, *The Taylor series*, Dover, New York, 1957, p. 396.

6. H. H. Körle, *Über Unsteige absolute Riesz-Summujung*. I, II, Math. Ann. **176** (1968), 45–52; ibid. **177** (1968), 230–234.

7. J. S. Ratti, *Tauberian theorems for absolute summability*, Proc. Amer. Math. Soc. **18** (1967), 775–781.

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