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CODIMENSION TWO SUBMANIFOLDS OF POSITIVE CURVATURE

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ABSTRACT. In this note it is proven that a compact connected n-dimensional Riemannian manifold of positive curvature, isometrically immersed in $(n+2)$-dimensional Euclidean space, is a homotopy sphere if $n > 3$; hence it is homeomorphic to a sphere if $n > 5$.

1. Introduction. Among the most beautiful applications of Morse theory are those which relate topology to curvature of Riemannian manifolds. We are interested in applying Morse theory to a slightly different problem: that of determining relationships between topology and curvature of submanifolds of low codimension in Euclidean space.

Let us recall some known facts. According to Alan Weinstein [**8**], an n-dimensional Riemannian manifold M^n with positive sectional curvatures, which is isometrically immersed in $(n + 2)$-dimensional Euclidean space E^{n+2}, must have positive curvature operators. This implies, according to a theorem of Daniel Meyer [**3**], that if in addition, M^n is compact and connected, its universal cover must be a rational homology sphere. In this note, we will use an inequality of Chang-shing Chen to show that in fact M^n must be a homotopy sphere:

THEOREM. *Let M^n be a compact connected n-dimensional Riemannian manifold with positive sectional curvatures, which is isometrically immersed in E^{n+2}. Then if $n \geqslant 3$, M^n is a homotopy sphere. Hence if $n \geqslant 5$, M^n is homeomorphic to a sphere.*

The last sentence follows from the rest of the theorem by means of the h-cobordism theorem [**5**, §9].

For example, our theorem implies that if M^n is a spherical space form which admits an isometric immersion in E^{n+2}, then M^n must be simply connected. (Compare [**6**, Propositions 4 and 5].)

2. The integrated Morse inequalities. We consider an immersion f from a compact n-dimensional manifold M^n into N-dimensional Euclidean space E^N. Let S^{N-1} denote the unit sphere at the origin of E^N. For each $u \in S^{N-1}$,

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we have a height function

$$h_u\colon M^n \to \mathbf{R} \quad \text{defined by} \quad h_u(p) = f(u)\cdot p,$$

where the dot denotes the usual Euclidean dot product. For almost all u, h_u is a Morse function. For each $u \in S^{N-1}$, let $\mu_k(u)$ denote the number of nondegenerate critical points of index k of the height function h_u. Following Kuiper [2], we consider the *total curvature of index* k,

$$\tau_k = \frac{1}{\operatorname{vol}(S^{N-1})} \int_{S^{N-1}} \mu_k(u).$$

We can regard τ_k as the "average number of critical points of index k." Since replacing u by $-u$ changes critical points of index k into critical points of index $n-k$, we see that $\tau_{n-k} = \tau_k$.

We will apply the Morse inequalities [4, §5] to the height functions h_u. Choose a field F and set $\beta_k = \dim H_k(M^n; F)$. If u is an element of S^{N-1} such that h_u is a Morse function, then

$$\beta_k \leqslant \mu_k(u), \quad \beta_1 - \beta_0 \leqslant \mu_1(u) - \mu_0(u), \ldots.$$

We average over S^{N-1} to obtain "Morse inequalities" which relate Betti numbers to total curvatures:

$$\beta_k \leqslant \tau_k, \quad \beta_1 - \beta_0 \leqslant \tau_1 - \tau_0, \ldots.$$

In particular, if M^n is connected, so that $\beta_0 = 1$,

$$\beta_1 \leqslant \tau_1 - \tau_0 + 1. \tag{1}$$

If M^n is orientable over F, then by Poincaré duality,

$$\beta_{n-1} \leqslant \tau_{n-1} - \tau_n + 1. \tag{2}$$

(1) and (2), together with $\beta_k \leqslant \tau_k$, yield

$$\beta_1 + \cdots + \beta_{n-1} \leqslant (\tau_1 + \cdots + \tau_{n-1}) - (\tau_0 + \tau_n) + 2. \tag{3}$$

3. Proof of the theorem. We will make use of an inequality of Chen, which holds when M^n has positive curvature and $N = n + 2$:

$$\tau_1 + \cdots + \tau_{n-1} < \tau_0 + \tau_n. \tag{4}$$

This inequality follows from Proposition 2.5 of [1]. (Note that due to a typographical error, the inequality as stated in [1] is reversed.) It follows immediately from (3) and (4) that

$$\beta_1 + \cdots + \beta_{n-1} < 2, \tag{5}$$

so long as M^n is orientable over the field F.

We now show that under the hypotheses of our theorem, M^n must be simply connected. If not, $\pi_1(M^n)$ contains a subgroup isomorphic to $\mathbf{Z}_p$ for some prime number p. Let $\tilde{M}^n$ be the Riemannian covering of M^n corresponding to this subgroup. It is easy to see that $\tilde{M}^n$ is compact and orientable over $\mathbf{Z}_p$. Moreover, the composition of the covering projection with the isometric immersion $M^n \to E^{n+2}$ yields an isometric immersion $\tilde{M}^n \to E^{n+2}$.

If we set $\beta_k = \dim H_k(\tilde{M}^n; \mathbf{Z}_p)$, we find that $\beta_1 = \beta_{n-1} = 1$, contradicting (5). Thus M^n must be simply connected.

Next we show that M^n has the same homology groups over the integers as a sphere. Since the work of Weinstein and Meyer shows that M^n is a rational homology sphere, we need only show that M^n has no torsion. But if $H_k(M^n; \mathbf{Z})$ contains torsion for some k, $0 < k < n$, then $H_k(M^n; \mathbf{Z}_p) \neq 0$ for some prime p, and by Poincaré duality $H_{n-k}(M^n; \mathbf{Z}_p) \neq 0$. This contradicts (5) unless $k = n/2$. Thus if M^n is not a homology sphere over the integers, it must be even-dimensional, say $n = 2m$, and all its torsion must lie in $H_m(M^n; \mathbf{Z})$. But again by Poincaré duality (torsion $H_m(M^n; \mathbf{Z}) \cong$ torsion $H_{m-1}(M^n; \mathbf{Z})$), this is impossible.

Thus M^n is simply connected and has the homology of a sphere, so by the Hurewicz isomorphism theorem, $\pi_i(M^n) = 0$ for $0 < i < n$ and $\pi_n(M^n) \cong \mathbf{Z}$. Let $g: S^n \to M^n$ be a representative for a generator of $\pi_n(M^n)$. g is a mapping between simply connected CW complexes which induces isomorphisms of homology groups. Hence by [7, p. 399, Theorem 9 and p. 405, Corollary 24], g is a homotopy equivalence and M^n is indeed a homotopy sphere. This finishes the proof of the theorem.

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