AN IDENTITY ON ALGEBRAS OVER A HOPF ALGEBRA

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Abstract. Let $A$ be a connected Hopf algebra which has an associative comultiplication $\psi: A \to A \otimes A$. Let $\chi: A \to A$ be the canonical conjugation on $A$. Let $M$ be a graded algebra over the Hopf algebra $A$. If $x, y \in M$, $\psi(a) = \Sigma a' \otimes a''$, then we have the identity

$$ax \cdot y = \Sigma (-1)^{\deg x \cdot \deg a''} a''(x \cdot \chi(a'') y).$$

If $X$ is a topological space and $x, y \in H^*(X; \mathbb{Z}_2)$, then we have the identity

$$\text{Sq}^n x \cdot y = \sum_{i=0}^{n} \text{Sq}^i \left( x \cdot \chi(\text{Sq}^{n-i}) y \right)$$

(for the proof see [1]). This formula has been proved useful in quite a few occasions (for example see [1], [2]).

In this note we give a generalization of this formula for algebras over a Hopf algebra.

Let $A$ be a Hopf algebra over a commutative ring with unit. We assume that $A$ is connected and that its comultiplication $\psi$ is associative. Let $\chi: A \to A$ be the canonical conjugation of $A$. Let $M$ be a graded algebra over the Hopf algebra $A$. For the terminology and the basic results, we refer to [3].

Under those assumptions, we have

Theorem. Let $x, y \in M$ and $\psi(a) = \Sigma a'_i \otimes a''_i$. Then we have

$$ax \cdot y = \Sigma (-1)^{\deg x \cdot \deg a''_i} a''_i(x \cdot \chi(a''_i) y).$$

Proof. Let

$$\psi(a'_i) = \sum_j b'_{ij} \otimes b''_{ij} \quad \text{and} \quad \psi(a''_i) = \sum_j c'_{ij} \otimes c''_{ij}.$$

Since $\psi$ is associative, we will have

$$\sum_{ij} b'_{ij} \otimes b''_{ij} \otimes a''_i = \sum_{ij} a'_{ij} \otimes c'_{ij} \otimes c''_{ij}.$$

From the basic properties of tensor products and the fact that $\chi$ is linear, we get that

$$\sum_{ij} b'_{ij} \otimes b''_{ij} \cdot \chi(a''_i) = \sum_{ij} a'_{ij} \otimes c'_{ij} \cdot \chi(c''_{ij}).$$

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So we get (see [1, p. 171])

\[
\sum_{i,j} (-1)^{\deg x \cdot \deg b_i^j} b_i^j \cdot x \cdot (-1)^{\deg x \cdot \deg \alpha_i^j} b_i^j \chi(\alpha_i^j)y
\]

\[
= \sum_i (-1)^{\deg x \cdot \deg \alpha_i^j} \alpha_i^j x \cdot \left( \sum_j c_i^j \chi(c_i^j) \right)y.
\]

But \(\sum_j c_i^j \chi(c_i^j) = 0\) unless \(\deg \alpha_i^j = 0\), so the second member of the previous identity is \(ax \cdot y\). Because of the "Cartan" formula (see [3, p. 173]), the first member of the previous identity is equal to the second member of the identity that we want to prove. That ends the proof.

REFERENCES


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