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AN IDENTITY ON ALGEBRAS OVER A HOPF ALGEBRA

STAVROS PAPASTAVRIDIS

ABSTRACT. Let A be a connected Hopf algebra which has an associative comultiplication $\psi: A \to A \otimes A$. Let $\chi: A \to A$ be the canonical conjugation on A. Let M be a graded algebra over the Hopf algebra A. If $x, y \in M$, $\psi(a) = \Sigma a' \otimes a''$, then we have the identity

$$ax \cdot y = \Sigma(-1)^{\deg x \cdot \deg a''} a'(x \cdot \chi(a'')y).$$

If X is a topological space and $x, y \in H^*(X; Z_2)$, then we have the identity

$$\mathrm{Sq}^n x \cdot y = \sum_{i=0}^{n} \mathrm{Sq}^i\left(x \cdot \chi(\mathrm{Sq}^{n-i})y\right)$$

(for the proof see [1]). This formula has been proved useful in quite a few occasions (for example see [1], [2]).

In this note we give a generalization of this formula for algebras over a Hopf algebra.

Let A be a Hopf algebra over a commutative ring with unit. We assume that A is connected and that its comultiplication ψ is associative. Let χ: $A \to A$ be the canonical conjugation of A. Let M be a graded algebra over the Hopf algebra A. For the terminology and the basic results, we refer to [3]. Under those assumptions, we have

THEOREM. *Let $x, y \in M$ and $\psi(a) = \Sigma_i a_i' \otimes a_i''$. Then we have*

$$ax \cdot y = \sum_i (-1)^{\deg x \cdot \deg a_i''} a_i'(x \cdot \chi(a_i'')y).$$

PROOF. Let

$$\psi(a_i') = \sum_j b_{ij}' \otimes b_{ij}'' \quad \text{and} \quad \psi(a_i'') = \sum_j c_{ij}' \otimes c_{ij}''.$$

Since ψ is associative, we will have

$$\sum_{i,j} b_{ij}' \otimes b_{ij}'' \otimes a_i'' = \sum_{i,j} a_{ij}' \otimes c_{ij}' \otimes c_{ij}''.$$

From the basic properties of tensor products and the fact that χ is linear, we get that

$$\sum_{i,j} b_{ij}' \otimes b_{ij}'' \cdot \chi(a_i'') = \sum_{i,j} a_i' \otimes c_{ij}' \cdot \chi(c_{ij}'').$$

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So we get (see [1, p. 171])

$$\sum_{i,j}(-1)^{\deg x\cdot\deg b''_{ij}}b'_{ij}x\cdot(-1)^{\deg x\cdot\deg a''_{ij}}b''_{ij}\chi(a''_i)y$$

$$=\sum_i(-1)^{\deg x\cdot\deg a''_i}a'_i x\cdot\left(\sum_j c'_{ij}\chi(c''_{ij})\right)y.$$

But $\sum_j c'_{ij}\chi(c''_{ij}) = 0$ unless $\deg a''_i = 0$, so the second member of the previous identity is $ax\cdot y$. Because of the "Cartan" formula (see [3, p. 173]), the first member of the previous identity is equal to the second member of the identity that we want to prove. That ends the proof.

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University of Athens, Mathematical Institute, Solonos 57, Athens 143, Greece

Current address: Department of Mathematics, University of Crete, Iraklion, Greece