A REMARK ON THE GROTHENDIECK RESIDUE MAP

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Abstract. The purpose of this note is to give a direct proof that a global integral over a compact complex manifold \( X \) can be evaluated on the zero set of a meromorphic vector field on \( X \) with isolated zeros via a Grothendieck residue morphism. A special case of this evaluation is the meromorphic vector field theorem of Baum and Bott [1]. The present proof suggests some complements of the M.V.F. Theorem which are contained in Theorem 2.

1. Introduction. Statement of results. Given a finite, nontrivial, but possibly unreduced subvariety \( Z \) of a connected compact complex manifold \( X \) of dimension \( n \), and given \( \omega \in H^n(X, \Omega^n) \), the global integral \( \int_X \omega \) can always be evaluated as a sum of residues (in the sense of Grothendieck) on \( Z \). The proofs of this fact in the literature are valid in algebraic geometry and are necessarily complicated, cf. [7], [13]. On the other hand, what we shall show is that if one makes the assumption that \( Z \) is the variety of zeros of a meromorphic vector field \( V \) on \( X \), then a reformulation of the fundamental commutative diagram of [13], which makes the contribution from \( Z \) explicit, can be proven simply (Theorem 1). Moreover this approach immediately suggests some complements to the Meromorphic Vector Field Theorem [1], [2], which are given in Theorem 2.

To state our results precisely, let \( T \) denote the holomorphic tangent bundle of \( X \), \( L \) a holomorphic line bundle on \( X \), and \( \Theta \) (resp. \( \mathcal{L} \)) the sheaf of germs of holomorphic sections of \( T \) (resp. \( L \)). Contraction by \( V \in H^0(X, T \otimes L) \) on holomorphic \( p \)-forms is an operator \( i(V) : \Omega^p \to \Omega^{p-1} \otimes \mathcal{L} \) on the sheaf level, which defines a sheaf of ideals \( I_Z = i(V)(\Omega^1 \otimes \mathcal{L}^{-1}) \) in \( \Omega^0 = \mathcal{O} \). The subvariety \( Z \) defined by \( I_Z \) is called the variety of zeros of \( V \). The structure sheaf of \( Z \) is by definition the sheaf of rings \( \mathcal{O}_Z = \mathcal{O} / I_Z \). For any sheaf \( \mathcal{F} \) of \( \mathcal{O} \)-modules on \( X \), let \( \mathcal{F}_Z = \mathcal{F} \otimes_{\mathcal{O}} \mathcal{O}_Z \). The Grothendieck Residue Map we will associate to \( V \) is a map \( \text{Res} : H^0(X, \mathcal{L}_Z^p) \to C \) about which we shall prove the following result. It will always be assumed that \( Z \) is finite but nontrivial.

**Theorem 1.** There exists a map \( m : H^0(X, \mathcal{L}_Z^p) \to H^n(X, \Omega^n) \) (depending only on \( V \)) such that

Received by the editors May 28, 1976 and, in revised form, April 20, 1977.


Supported in part by a grant from the National Research Council.

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\[
H^0(X, \Omega^n) \xrightarrow{\text{Res}} C \\
(-1)^m \downarrow \text{tr} \\
H^n(X, \Omega^n)
\]

commutes, where \(\text{tr}\) is the map \((1/2\pi i)^n f_X\) (cf. [13]).

Given an ad-invariant linear map \(p: \text{gl}(n, C)^{\otimes k} \to C\), there is a natural element \(p(V_0)\) of \(H^0(X, \mathcal{L}_2^n)\) such that in the case \(k = n\), \(\text{tr}(m(p(V_0)))\) is a characteristic number of the virtual bundle \(T - L^{-1}\). The assertion \((-1)^n \text{tr}(m(p(V_0))) = \text{Res} p(V_0)\) is of course the M.V.F. Theorem. In §5, we will show that in addition one can use (1) to prove

**THEOREM 2.** Suppose \(\deg p = k < n\) and that \(\sigma \in H^0(X, L)\). Let \(\sigma^{n-k-\bar{p}}(V_0)\) denote the image of \(\sigma^{n-k} \otimes p(V_0)\) under the natural pairing \(H^0(X, L^{n-k}) \otimes H^0(X, \mathcal{L}_2^n) \to H^0(X, \mathcal{L}_2^n)\). Then \(\text{Res} \sigma^{n-k-\bar{p}}(V_0) = 0\).

To prove Theorem 1, we employ a double complex with differentials \(i(V)\) and \(\delta\) to compute \(H^0(X, \mathcal{L}_2^n)\). The mapping \(m\) is an edge morphism in this double complex. (1) is a consequence of the projector trick of Bott [3] and an interesting local integral representation formula for \(\text{Res}\) (Lemma 4), which simply amounts to combining a local I.R.F. for the partial derivatives of a holomorphic function at a point with the computational algorithm for the local residue involving the Nullstellensatz.

**2. A double complex.** For a fixed integer \(m\), let \(A^{p,q}(L^m)\) denote \(C^\infty\), \(L^m\)-valued forms on \(X\) of type \((p, q)\). The operators \(i(V): A^{p,q}(L^m) \to A^{p-1,q}(L^{m+1})\) and \(\delta: A^{p,q}(L^m) \to A^{p+1,q}(L^m)\) satisfy \(i(V)^2 = \delta^2 = \delta i(V) + i(V)\delta = 0\), so \(D = i(V) + \delta\) is a total differential for the complex \(C^m = \Sigma_p C_{m-p}^{p,q}\) formed from the double complex \(\{C^m, i(V), \delta\}\).

**LEMMA 1.** Assuming that \(Z\) is finite, then \(H^0(C^m) \simeq H^0(X, \mathcal{L}_2^n)\).

Let \(\mathcal{E}^{p,q}(\mathcal{L}^k)\) denote the sheaf of germs of \(C^\infty\), \(L^k\)-valued \((p, q)\) forms on \(X\). To prove Lemma 1, we need

**LEMMA 2.** For any \(k\) we have a fine resolution of \(\mathcal{L}^k_Z\):

\[
0 \to \mathcal{L}^k_Z \xrightarrow{h} \mathcal{E}^{0,0}(\mathcal{L}^k) / i(V) \mathcal{E}^{1,0}(\mathcal{L}^k) \xrightarrow{\delta} \mathcal{E}^{0,1}(\mathcal{L}^k) / i(V) \mathcal{E}^{1,1}(\mathcal{L}^k) \xrightarrow{\delta} \cdots \xrightarrow{\delta} \mathcal{E}^{0,n}(\mathcal{L}^k) / i(V) \mathcal{E}^{1,n}(\mathcal{L}^k) \to 0
\]

where \(h\) is the natural inclusion.

**PROOF.** One first notes that for any \(k\) the Dolbeault resolution

\[
0 \to \mathcal{L}^k \to \mathcal{E}^{0,0}(\mathcal{L}^k) \xrightarrow{\delta} \mathcal{E}^{0,1}(\mathcal{L}^k) \xrightarrow{\delta} \mathcal{E}^{0,n}(\mathcal{L}^k) \to 0
\]

is an exact sequence of \(\mathcal{O}\)-modules from which (2) is obtained via tensorisation by \(\mathcal{O}_Z\). Lemma 2 follows from the fact that \(\mathcal{E}^{p,q}(\mathcal{L}^k)\) is a flat \(\mathcal{O}\)-module,
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This lemma does not assume \( Z \) is finite. Finiteness of \( Z \) is used to conclude exactness of

\[
0 \to \Omega^n \otimes \mathcal{L}^k - n + i(V) \Omega^{n-1} \otimes \mathcal{L}^{k-n+1} \to \cdots \to \Omega^1 \otimes \mathcal{L}^{k-1} - n + i(V) \mathcal{L}^k.
\]  
(3)

To prove Lemma 1, consider the exact sequence obtained from (2):

\[
0 \to H^0(X, \mathcal{L}_Z^0) \xrightarrow{h} C_k^{0.0} / i(V)C_k^{1.0} \xrightarrow{\delta} C_k^{1.1} / i(V)C_k^{1.1}.
\]  
(4)

For \( s \in H^0(X, \mathcal{L}_Z^2) \), choose \( s_0 \in C_k^{0.0} \) such that \( h(s) = s_0 \) modulo \( i(V)C_k^{1.1} \). By (4), \( \delta s_0 = i(V)s_1 \) for some \( s_1 \in C_k^{1.1} \). Since \( i(V)\delta s_1 = -\delta i(V)s_1 = \delta^2 x_0 = 0 \), there exists by (3) an \( s_2 \in C_k^{1.2} \) so that \( \delta s_1 = i(V)s_2 \). Continuing in this manner, one gets a total cocycle \( S = s_0 - s_1 + \cdots + (-1)^{n}s_n \) of \( C_k \) of degree 0, and the map \( s \to S \) induces a morphism \( \phi_k : H^0(C_k^0) \to H^0(C_k^1) \).

To show that \( \phi_k \) is an isomorphism, we produce its inverse. Let \( S \) be a total cocycle in \( C_k^0 \) denoted as above. Then \( \delta s_0 = i(V)C_k^{1.1} \). Hence by (4), there exists a unique \( s \in H^0(X, \mathcal{L}_Z^2) \) such that \( h(s) = s_0 \mod i(V)C_k^{1.0} \). Then \( S \to s \) induces the inverse of \( \phi_k \), and the proof of Lemma 1 is complete.

One now defines \( m : H^0(X, \mathcal{L}_Z^n) \to H^n(X, \mathcal{L}^n) \) \((n = \dim X)\) by composing \( \phi_k^{-1} \) with the natural edge morphism \( H^0(C_k^n) \to H^0(X, \mathcal{L}^n) \) induced by mapping \( S \) in \( C_k^0 \) to the component \( s_n \) of \( S \) in \( A^n'(L^0) \).

3. The morphism Res. Let \( U \) be an open ball about the origin \( O \) of \( \mathbb{C}^n \), which is the only common zero of \( a_1, \ldots, a_n \in H^0(U, \mathcal{O}) \). The local residue at \( O \) of \( \omega \in H^0(U, \mathcal{L}^n) \) with respect to \( a_1, \ldots, a_n \) is defined in [11] as

\[
\text{Res}_a \left( a_1 \cdots a_n, \omega \right) = \left( \frac{1}{2\pi i} \right)^n \int_{\partial D} \cdots \times \partial D \frac{\omega}{a_1 \cdots a_n}
\]  
(5)

where \( D \) is a disc about \( O \) in \( C \) chosen so that \( \partial D \times \cdots \times \partial D \) misses the hypersurface \( \{ z : (a_1 \cdots a_n)(z) = 0 \} \) in \( U \). The local residue defined by (5) coincides with the Grothendieck residue, and hence can be computed by the well-known algorithm [1], [2], [7], [11], [13]. If \( a_1, \ldots, a_n \in H^0(U, \mathcal{O}) \), then

\[
\text{Res} \left( a_1 \cdots a_n \right)
\]

is unambiguously defined for all \( \omega \in H^0(U, \mathcal{L}^n \otimes \mathbb{C}^n) \).

Suppose \( V \in H^0(X, T \otimes L) \) has isolated zeros \( Z \), and let \( (z_1, \ldots, z_n) \) denote local coordinates for \( X \) near \( \xi \in Z \) with \( z_i(\xi) = 0 \) for each \( i \). One may locally express \( V = \Sigma a_i \partial / \partial z_i \), where \( a_1, \ldots, a_n \) are local sections of \( L \) whose only common zero is \( \xi \). Given \( s \in \mathcal{L}_Z^n \), the local residue of \( s \) at \( \xi \) is defined to be

\[
\text{Res}(s)_\xi = \text{Res} \left( s \frac{dz_1}{a_1} \cdots \frac{dz_n}{a_n} \right).
\]  
(6)

Under change of coordinates, Res transforms in a manner so as to imply that (6) depends only on \( V \) and \( s \). In fact, \( \text{Res}(s)_\xi = 0 \) if \( s \in (I_\xi \mathcal{L}^n)_\xi \) (by [7]), and consequently Res is defined as a morphism \( \text{Res}_\xi : \mathcal{L}_Z^n \to C \). Define Res:

\( H^0(X, \mathcal{L}_Z^n) \to C \) by Res = \( \Sigma Z \text{Res}_\xi \).
4. The proof of Theorem 1. In order to localize \( \omega = m(S) \in H^n(X, \Omega^n) \), we will use Bott's projector trick. Recall that projector for \( V \) is an \( L^{-1} \) valued \((1,0)\) form \( \pi \) on \( X - Z \) such that \( \pi(V) = 1 \). Bott's key observation concerning projectors can be rephrased in this context as the observation that the differential form

\[
\tau = \pi \left( s_0 (\bar{\partial} \pi)^{-1} - s_1 (\bar{\partial} \pi)^{-2} + \cdots + (-1)^{n-1} s_{n-1} \right)
\]  

(7)
on \( X - Z \) satisfies \( s_n = (-1)^{n-1} \partial \tau \) provided \( S = s_0 + s_1 + \cdots + s_n \) is a total cocycle of \( C^n_0 \). If \( \{ W_i \} \) is a finite covering of \( Z \) by disjoint coordinate balls such that \( W_i \cap Z = \{ \xi_i \} \), then by Stokes' Theorem

\[
\int_X s_n = (-1)^n \sum \int_{\partial W_i} \tau.
\]  

(8)

Now the right-hand side of (8) can be vastly simplified by the following observation.

**Lemma 3.** For any \( S \in H^0(C) \), there is a representing cocycle \( S = s_0 + \cdots + s_n \) such that \( s_0 \) is holomorphic near \( Z \), and, if \( i > 0 \), then \( s_i = 0 \) near \( Z \).

**Proof.** \( s_0 \) can obviously be so chosen. The existence of the \( s_i \) follows from repeated application of (3).

The proof of Theorem 1 now results from the following local integral representation formula for \( \text{Res} \).

**Lemma 4.** For any projector \( \pi \) for \( V \) and for any \( \xi \in Z \),

\[
\text{Res}(s)_{\xi} = \left( \frac{1}{2\pi i} \right)^n \int_{\partial W} s \pi (\bar{\partial} \pi)^{-1}
\]  

(9)

where \( s \in \mathcal{L}^n_{\xi} \) and \( W \) is a sufficiently small ball centered at \( \xi \).

**Proof.** Since \( Z \) is finite, \( \mathcal{L}^n_{\xi} \subset H^0(X, \mathcal{L}^n) \). Therefore, \( m(s) \) is defined and \( \int_X m(s) = \int_{\partial W} s \pi (\bar{\partial} \pi)^{-1} \). Hence (9) is independent of \( \pi \). Let \( \pi \) be the projector for \( V \) defined in a neighborhood \( W \) of \( \xi \) as follows. Let \( V = \Sigma a_i \partial / \partial z_i \), on \( W \) as above. By Hilbert's Nullstellensatz, there exist positive integers \( \alpha_1, \ldots, \alpha_n \) and \( b_{ij} \in H^0(W, L^{-1}) \) such that \( z_i^{\alpha_i} = \Sigma b_{ij} a_j \). Set

\[
\pi = u^{-1} \Sigma z_i^{\alpha_i} b_{ij} dz_j,
\]

where \( u = \Sigma (z_i z_i)^{\alpha} \). Note

\[
\tau(\bar{\partial} \pi)^{-1} = (n - 1)! \left( (-1)^{n(n-1)/2} u^{-n} \det \| b_{ij} \| \right)
\]

\[
\times \left( \sum (-1)^{i-1} z_i^{\alpha_i} dz_1^\alpha_1 \wedge \cdots \wedge dz_i^\alpha_i \wedge d\bar{z}_1^\alpha_1 \wedge \cdots \wedge d\bar{z}_n^\alpha_n \right)
\]

Now, by applying a standard I.R.F. for the partial derivatives of a holomorphic function \( g \) defined in a neighbourhood of \( \overline{W} \) [10, p. 56], one gets

\[
\left( \frac{1}{2\pi i} \right)^n \int_{\partial W} g \sigma (\bar{\partial} \pi)^{-1} = 1 / (\alpha - 1)! D^{\alpha} (\det \| b_{ij} \| g)(\xi)
\]  

(10)

where \( \alpha \) and \( \alpha - 1 \) denote, respectively, the multi-indices \((\alpha_1, \ldots, \alpha_n)\) and \((\alpha_1 - 1, \ldots, \alpha_n - 1)\). Lemma 4 follows from the fact that the r.h.s. of (10) is
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\[ \text{Res}_i \left( \frac{g \, dz_1 \cdots \, dz_n}{a_1, \ldots, a_n} \right), \]

by the algorithm.

To finish the proof of (1), let \( s \in H^0(X, \mathcal{E}_Z^n) \) and let \( S = s_0 + s_1 + \cdots + s_n \) be a total cocycle of \( C^n_0 \) representing \( \phi(s) \) satisfying all the conclusions of Lemma 3. Then \((-1)^n m(S) = (-1)^n s_n\), so

\[ \left( -\frac{1}{2\pi i} \right)^n \int_X s_n = \left( \frac{1}{2\pi i} \right)^n \sum_{\partial W_j} s_0 \pi(\tilde{\pi})^{n-1} = \text{Res}(s) \]

by (9) and Lemma 4.

5. Some residue formulas. If \( V \) is a global section of the tangent sheaf \( \Theta \) of \( X \) (i.e. a holomorphic vector field on \( X \)), then the Lie bracket \( Y \to [V, Y] \) induces a \( C \)-linear map \( \tilde{V}: \Theta \to \Theta \) lifting the derivation \( \tilde{V}: \Theta \to \Theta \), i.e. \( \tilde{V}(fY) = V(f)Y + f\tilde{V}(Y) \). Any sheaf \( \mathcal{F} \) of \( \Theta \)-modules admitting such a lifting \( \tilde{V} \) is called \( \Theta \)-equivariant and \( \tilde{V} \) is called an equivariant lift of \( V \). Note that \( \tilde{V} \) defines \( V_0 \in H^0(X, \text{Hom}_0(\Theta, \Theta)_Z) \). Analogously, an equivariant lift \( \tilde{V}: \mathcal{F} \to \mathcal{F} \otimes L \) of \( V \in H^0(X, T \otimes L) \) defines an element \( V_0 \in H^0(X, (\text{Hom}_0(\mathcal{F}, \Theta)) \otimes \mathcal{L})_Z \). Now \( \Theta \) is generally not equivariant for an arbitrary \( V \in H^0(X, T \otimes L) \), however \( \tilde{V} \) admits a well-defined localization \( V_0 \in H^0(X, (\text{Hom}_0(\Theta, \Theta)) \otimes \mathcal{L})_Z \). In fact, choose a covering \( \{ U_a \} \) of \( X \) such that both \( L \mid U_a \) and \( T \mid U_a \) are locally trivial for each \( a \), and on \( U_a \) write \( V = w_a \otimes t_a \) where \( w_a \in H^0(U_a, \Theta) \), \( t_a \in H^0(U_a, \mathcal{L}) \), and \( t_a \) is nowhere vanishing. For \( Y \in H^0(U_a, \Theta) \), set \( \tilde{V}_a(Y) = [w_a, Y] \otimes t_a \). Note that on \( U_a \cap U_b \), \( \tilde{V}_a(Y) - \tilde{V}_b(Y) = t_b^{-1}i(Y)d(t_b^{-1}) \otimes V \), consequently the \( \tilde{V}_a \) patch on \( Z \) giving \( V_0 \) as asserted.

Let \( p(V_0) \) denote the element of \( H^0(X, \mathcal{E}_Z^n) \) obtained by applying the ad-invariant symmetric linear map \( p: \text{gl}(n, C) \otimes k \to C \) to \( V_0 \). In order to construct the class \( \phi_k(p(V_0)) \) in \( H^0(C_k^2) \), first choose a local holomorphic connection \( D_a \) for \( T \mid U_a \), and let \( D = \sum \rho_a D_a \) be the connection of type (1, 0) on \( T \) where \( \rho_a \) is a partition of unity fitted to \( \{ U_a \} \). Following [1], consider the \( \Theta \)-linear map \( \psi_\#: \Omega^1 \to \text{Hom}(\Theta, \Theta) \otimes \Omega^1 \) so that if \( \omega \in \Omega^1 \), and \( Y = \Theta \), then \( i(Y)\psi_\# \omega = \omega(w)Y \). Let \( \Gamma \in A^{0,0}(\text{Hom}(T, T)) \) be given by \( \Gamma = \sum \rho_a \{ \psi_\#(\tau^{-1}) - D_a \} \), and let \( K^* = \tilde{\Gamma} \in A^{1,1}(\text{Hom}(T, T)) \). Finally, define \( \tau = \sum \rho_a \{ V_a - i(V)D_a \} \in A^{0,1}(\text{Hom}(T, T) \otimes L) \). The following lemma is proved by a local calculation which will be omitted.

**Lemma 5.** \( i(V)K^* + \tilde{\tau} = 0 \) in \( A^{0,1}(\text{Hom}(T, T) \otimes L) \).

One may therefore perform the ad-invariant map construction to get a class \( p(K^* + \tau) = p((K^* + \tau)^{\otimes k}) \) in \( C_k^0 \) where \( k = \text{deg} \, p \). Because of Lemma 5, it follows that \( p(K^* + \tau) \) is a total cocycle. Note

\[ p(K^* + \tau) = p(K^*^{\otimes k}) + kp(K^*^{\otimes k-1} \otimes \tau) + \cdots + p(\tau^{\otimes k}). \]

Note that by definition, \( p(\tau^{\otimes k}) \) is an extension of \( p(V_0) \) to \( X \). If \( k = \text{deg} \, p = \)
\( n \), we therefore get by (1) that
\[
\left( \frac{1}{2\pi i} \right)^n \int_X p(K^{\otimes n}) = \text{Res } p(V_0)
\]
which is part of the M.V.F. Theorem. The rest of the theorem is an identification of \((1/2\pi i)^n \int_X p(K^{\otimes n})\) with a characteristic number of \( T - L^{-1} \).

In order to prove Theorem 2, note that in the case \( \deg p = k < n \), if \( \sigma \in H^0(X, L) \), then \( \sigma^{n-k}p(V_0) \) lies in \( H^0(X, L^k) \) and \( m_\phi(\sigma^{n-k}p(V_0)) = (-1)^n m(\sigma^{n-k} \times p(K^{\otimes k})) \) since \( \phi_\sigma(\sigma^{n-k}p(V_0)) \) is represented by the total cocycle \( \sigma^{n-k}p(K\ast + \tau) \) of \( C^n \) (due to the fact that \( \sigma^{n-k} \) commutes with both \( i(V) \) and \( \partial \)). But clearly \( m(\sigma^{n-k}p(K\ast + \tau)) = 0 \) since \( \deg p < n \). Consequently \( \text{Res}(\sigma^{n-k}p(V_0)) = 0 \), by (1), as asserted.

**Remark.** The notion of \( V \)-equivariance is studied in [1] and in [4] from a different viewpoint. In [4] it is shown that if \( X \) is projective, then given a holomorphic vector bundle \( E \) on \( X \), there exists a \( V \) \in \( H^0(X, \mathcal{T} \otimes L) \) with isolated zeros for which \( E \) is \( V \) equivariant. Thus \( E \)'s characteristic numbers can be computed as above.

**References**

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