SOME APPLICATIONS OF THE STONE-WEIERSTRASS THEOREM

P. J. DE PAEPE

Abstract. Let $A$ be a function algebra on a compact Hausdorff space $X$ and let $f \in A$. The Stone-Weierstrass theorem is used to obtain results on the function algebra on $X$ generated by the elements of $A$ and the function $f$.

Preliminaries. If $A$ is a collection of continuous functions on a compact Hausdorff space $X$ separating the points of $X$, $[A; X]$ denotes the function algebra on $X$ generated by the elements of $A$, i.e., the smallest algebra of continuous functions on $X$ containing the elements of $A$ and the constant functions which is uniformly closed in $C(X)$, i.e., closed in the algebra $C(X)$ of all continuous functions on $X$ provided with the supremum norm $||f||_X = \sup\{|f(x)|: x \in X\}$. If $f$ (respectively $A$) is a function (a collection of functions) on $X$, and $Y$ is a subset of $X$, then $f|Y$ ($A|Y$) denotes the restriction of $f$ to $Y$ (the collection of restrictions of elements of $A$ to $Y$).

If $A$ is a function algebra on $X$, a closed subset $K$ of $X$ is called a peak set for $A$ if there exists a function $f \in A$ such that $f(K) = \{1\}$ and $|f(x)| < 1$ for all $x \in X \setminus K$. If $K$ is a peak set for $A$ then $A|K$ is closed in $C(K)$ [3, p. 163].

A subset $K$ of $X$ is called a set of antisymmetry for $A$ [2, p. 60] if any $f \in A$ which is real-valued on $K$ is constant on $K$. The collection of maximal sets of antisymmetry is a closed pairwise disjoint cover of $X$. The generalized Stone-Weierstrass theorem reads:

If $f \in C(X)$ and $f|K \in A|K$ for all maximal sets of antisymmetry $K$ for $A$, then $f \in A$.

If $X$ is a compact subset of the complex plane, $P(X)$ is the function algebra on $X$ consisting of uniform limits on $X$ of polynomials; $R(X)$ is the function algebra on $X$ consisting of uniform limits on $X$ of rational functions with pole sets missing $X$.

The algebra $A(X)$ is the function algebra on $X$ consisting of all continuous functions on $X$ which are holomorphic on the interior of $X$.

Finally the complex homomorphism space of a function algebra $A$ is denoted by $\Delta A$. We abbreviate for $Y$ closed in $\Delta A : [A|Y] = [A|Y; Y]$.

In the following we determine the function algebra $[A, f; X]$ where $A$ is a
function algebra on $X$ and $f$ is a real-valued function on $X$ or the complex conjugate of an element of $A$. As corollaries we obtain results of Mergelyan, Minsker and Preskenis.

**Theorem 1.** Let $A$ be a function algebra on $X$, $f$ a real-valued continuous function on $X$. Let $X_\alpha = \{ x \in X : f(x) = \alpha \}$, $\alpha \in f(X)$. Then $[A, f; X] = \{ g \in C(X) : g|X_\alpha \in [A|X_\alpha] \text{ for each } \alpha \in f(X) \}$.

**Proof.** The inclusion "$\subseteq"$ is trivial. Conversely, let $K$ be a maximal set of antisymmetry for $[A, f; X]$, then $K \subset X_{\alpha_0}$ for some $\alpha_0$. Any $g \in C(X)$ such that $g|X_\alpha \in [A|X_\alpha]$ for each $\alpha$ has the property that $g|K \in [A|K] \subseteq [A, f; X]|K$. The last equality follows from the fact that $X_{\alpha_0}$ is a peak set for $[A, f; X]$. So also $g|K \in [A, f; X]|K$, so by the Stone-Weierstrass theorem $g \in [A, f; X]$.

An immediate consequence is the following result of Mergelyan [4].

**Theorem 2 (Mergelyan).** Let $X$ be a compact subset of $\mathbb{C}$; $f$ a real-valued continuous function on $X$ such that $X_\alpha = \{ x \in X : f(x) = \alpha \}$ is polynomially convex for each $\alpha \in f(X)$. Then $[z, f; X] = [R(X), f; X]$ and this function algebra consists of all elements of $C(X)$ which are holomorphic on the interior of the sets $X_\alpha$.

**Proof.** Let $A = P(X)$ and apply Theorem 1. By the classical Mergelyan theorem $[A|X_\alpha] = P(X_\alpha) = A(X_\alpha)$. Trivially $[z, f; X] \subseteq [R(X), f; X] \subseteq \{ g \in C(X) : g \text{ is holomorphic on the interior of the sets } X_\alpha \}$.

**Theorem 3.** Let $A$ be a function algebra on $X$, $f \in A$. Then $[A, \tilde{f}; X] = \{ g \in C(X) : g|X_\alpha \in [A|X_\alpha] \text{ for each } \alpha \in f(X) \}$, where $X_\alpha = \{ x \in X : f(x) = \alpha \}$, $\alpha \in f(X)$.

**Proof.** Again the inclusion "$\subseteq$" is trivial. Since both $\text{Re} f$ and $\text{Im} f$ belong to $[A, \tilde{f}; X]$, $X_\alpha$ is the intersection of the peak sets for $[A, \tilde{f}; X] : \{ x \in X : \text{Re} f(x) = \text{Re} \alpha \}$, $\{ x \in X : \text{Im} f(x) = \text{Im} \alpha \}$, so is itself a peak set for $[A, \tilde{f}; X]$. Also if $K$ is a maximal set of antisymmetry for $[A, \tilde{f}; X]$, $K$ is contained in some $X_\alpha$ (since both $\text{Re} f$ and $\text{Im} f$ are constant on $K$). The conclusion now follows as in the proof of Theorem 1.

Applying this result to the algebra $R(X)$, $X \subset \mathbb{C}$, we obtain

**Theorem 4.** Let $X$ be a compact subset of the complex plane.

(i) Let $g \in R(X)$ and let $X_\alpha = \{ x \in X : g(x) = \alpha \}$, $\alpha \in g(X)$. Then $[R(X), \tilde{g}; X] = \{ f \in C(X) : f|X_\alpha \in R(X_\alpha) \text{ for each } \alpha \in g(X) \}$.

(ii) Let $g \in R(X)$ such that the level sets $X_\alpha$ of $g$ are polynomially convex. Then $[R(X), \tilde{g}; X]$ consists of all elements of $C(X)$ which are holomorphic on the interior of the sets $X_\alpha$.

In particular:

(iii) If $X$ is polynomially convex and $g \in P(X)$ then $[z, \tilde{g}; X]$ consists of all elements of $C(X)$ which are holomorphic on the interior of the level sets of $g$.

**Proof.** For a compact subset $Y$ of $X$, $[R(X)|Y] = R(Y)$ if and only if $Y$ is
APPLICATIONS OF THE STONE-WEIERSTRASS THEOREM

\( R(X) \)-convex. Since \( X_a \) is \( R(X) \)-convex \( [R(X)]_a = R(X_a) \). Now apply Theorem 3 and (i) follows. Using the classical Mergelyan theorem (ii) and (iii) follow immediately.

**Theorem 5.** Let \( X \) be a compact set of the complex plane.

(i) Let \( g \in R(X) \) such that the level sets \( X_a \) of \( g \) are polynomially convex and such that \( g \) is not constant on any of the components of the interior of \( X \). Then \( [R(X), \tilde{g}; X] = C(X) \).

In particular (Preskenis [6]):

(ii) If \( X \) is polynomially convex and \( g \in P(X) \) such that \( g \) is not constant on any of the components of the interior of \( X \) then \( [z, \tilde{g}; X] = C(X) \).

This result follows immediately from Theorem 4 since the hypothesis on \( g \) implies that the interior of the sets \( X_a \) are empty. An abstract version of Theorem 5(ii) is

**Theorem 6.** Let \( A \) be a function algebra on \( X \). Let \( f \in A \) and let \( Y \) be the polynomially convex hull of \( f(X) \). Let \( g \in P(Y) \) such that \( g \) is not constant on any of the components of the interior of \( Y \). Then \( [A, f; X] = [A, \tilde{g} \circ f; X] \).

**Proof.** Since \( g \in C(Y) \), \( \tilde{g} \) is uniformly approximable on \( Y \) by polynomials in \( z \) and \( \bar{z} \), so the inclusion "\( \supset \)" follows since \( \tilde{g} \circ f \in [A, \tilde{f}; X] \). Conversely, by Theorem 5(ii) it follows that \( [z, \tilde{g}; Y] = C(Y) \), hence \( \bar{z} \in [z, \tilde{g}; Y] \) so \( \tilde{f} \in [A, \tilde{g} \circ f; X] \).

Applying Theorem 5(ii) we obtain a result of Minsker [5]:

**Corollary 1 (Minsker).** Let \( X \) be a compact subset of the complex plane and \( m \in \mathbb{N} \). Then \( [z, \bar{z}^m; X] = C(X) \).

**Corollary 2.** Let \( f \) and \( g \) be holomorphic on a neighborhood of \( 0 \in \mathbb{C} \) such that \( \partial f / \partial z(0) \neq 0 \) and \( g \) is not constant near \( 0 \). Then there exists a closed disc \( D \) centered at the origin such that \( [f, \tilde{g}; D] = [g, \tilde{f}; D] = C(D) \).

**Proof.** Without loss of generality \( f(0) = 0 \), so if \( \delta > 0 \) is small enough \( z \) can be approximated uniformly on \( D = \{ |z| < \delta \} \) by polynomials in \( f \). Moreover we may assume that \( g \) is defined on \( D \). By Theorem 5(ii) \( [z, \tilde{g}; D] = C(D) \), so \( [f, \tilde{g}; D] = C(D) \). And \( [g, \tilde{f}; D] = [g, \bar{z}; D] \) consists of the complex conjugates of the elements of \( [\tilde{g}, z; D] \). So \( [g, \tilde{f}; D] = C(D) \).

**Corollary 3.** Let \( X = \{ 1 < |z| < 2 \} \subset \mathbb{C} \) and let \( r_1, r_2 \in R(X) \) such that \( r_1 \notin P(X) \) and \( r_2 \) is not constant on \( X \). Then \( [z, r_1, \bar{r}_2; X] = C(X) \).

**Proof.** Since \( P(X) \) is maximal in \( R(X) \) [1], \( [z, r_1; X] = R(X) \) and by Theorem 5(i) \( [R(X), \bar{r}_2; X] = C(X) \).

Using techniques similar to those in the previous section we prove two other results.

**Theorem 7.** Let \( n, m \in \mathbb{N} \) such that \( \gcd(n, m) = 1 \) and let \( D = \{ |z| < 1 \} \). If \( f \) is a nonconstant element of \( P(D) \), then \( [z^n, z^m, \tilde{f}; D] = C(D) \).
Proof. There exists $N \in \mathbb{N}$ such that $z^k \in A = [z^n, z^m; D]$ for each $k > N$. Without loss of generality we may assume $f(0) = 0$, so $f^N$ is uniformly approximable on $D$ by elements of $A$.

Therefore $\text{Re} f^N$ and $\text{Im} f^N$ belong to $[A, \tilde{f}; D]$. Let $K$ be a maximal set of antisymmetry for $[A, \tilde{f}; D]$. Then $K$ is contained in a level set $L$ of $f^N$ which is a peak set for $[A, \tilde{f}; D]$. Since $f^N$ is not constant on $D$, $L = \mathcal{P} \cup \{a_1, a_2, a_3, \ldots \}$ where $\mathcal{P}$ is a proper subset of $\{z = 1\}$ and where $\{a_1, a_2, a_3, \ldots \}$ is a discrete subset of $\{z < 1\}$. Now $\Delta A = D$. (Indeed: let $\phi \in \Delta A$ such that $\phi(z^N) = 0$. If $k \in \mathbb{N}$ such that $z^k \in A$ then $(\phi(z^k))^N (\phi(z^N))^k = 0$, so $\phi(z^k) = 0$ hence $\phi$ is point evaluation at the point 0. If $\phi(z^N) \neq 0$, let $a = \phi(z^{N+1})/\phi(z^N)$. For $k \in \mathbb{N}$ such that $z^k \in A$ we have $\phi(z^k) \cdot \phi(z^{Nk}) = \phi(z^{(N+1)k})$ so $\phi(z^k) = a^k$, hence $\phi$ is point evaluation at $a \in D$.)

Since $a_n$ is an isolated point of the peak set $L$, by Rossi's local peak set theorem [2, p. 91] there exists $f_n \in [A, \tilde{f}; D]$ such that $f_n(x) = 1$ for each $x \in L \setminus \{a_n\}$ and such that $|f_n(x)| < 1$ for each $x \in \{a_n\} \cup D \setminus L$. Since $[A, \tilde{f}; D]|L$ is closed we may assume $f_n(a_n) = 0$ [3, p. 164]. So the maximal set $K$ of antisymmetry reduces to a single point $\{a_n\}$ or else $K \subseteq \mathcal{P}$.

Now $\mathcal{P} = L \cap \cap_{n=1}^{\infty} \{x \in D : f_n(x) = 1\}$ is again a peak set for $[A, \tilde{f}; D]$ and is convex relative to the algebra $A$ (since $L$ is). So $\Delta[A, \tilde{f}; D]|P = \Delta[A]|P = P$. Since $z^n$ and $z^m$ have no zeros on $P$, $z^{-n}, z^{-m} \in [[A, \tilde{f}; D]|P]$, so $[[A, \tilde{f}; D]|P] = C(P)$ and since $[A, \tilde{f}; D]|P$ is closed in $C(P)$ we have $[A, \tilde{f}; D]|K = C(K)$. By the Stone-Weierstrass theorem it follows that $[A, \tilde{f}; D] = C(D)$.

Theorem 8 (Minsker [5]). Let $n, m \in \mathbb{N}$ such that $\gcd(n, m) = 1$ and let $X$ be a compact subset of $C$. Then $[z^n, z^m; X] = C(X)$.

Proof. $\text{Re} z^m$ and $\text{Im} z^m$ belong to $A = [z^n, z^m; X]$, so a maximal set of antisymmetry for $A$ consists of a finite number of points, so has to be a singleton. By the Stone-Weierstrass theorem $A = C(X)$.

We conclude with two questions. Let $D = \{z < 1\}$ and suppose that $f \in P(D)$ and $z^2$ separate the points of $D$. What can be said about the algebra $[z^2, \tilde{f}; D]$?

Let $X = \{1 \leq |z| < 2\}$. If $f \in R(X)$ and $z^2$ separate points of $X$, determine $[z^2, z^{-2}, \tilde{f}; X]$.

References


Instituut voor Propedeutische Wiskunde, Universiteit van Amsterdam, Amsterdam, The Netherlands