ZONES OF UNIFORM DECOMPOSITION IN TENSOR PRODUCTS

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Abstract. Let $V_{\lambda}$ be a finite dimensional irreducible module for a complex semisimple Lie algebra. It is shown that the decomposition of tensor products $V_{\lambda} \otimes V_{\tau}$ for all dominant integral weights $\tau$ may be derived from those for a finite set of such $\tau$. An explicit choice of such a finite set (depending on $\lambda$) is given.

Introduction. Let $L$ be a complex semisimple Lie algebra with simple roots \{\(\alpha_1, \ldots, \alpha_l\)\} and fundamental weights \{\(\omega_1, \ldots, \omega_l\)\}. That is, \(\{\omega_1, \ldots, \omega_l\}\) is a basis of the integral weight lattice, $\Lambda$, such that $\langle \omega_i, \alpha_j \rangle = 2(\omega_i, \alpha_j)/(\alpha_j, \alpha_j) = \delta_{ij}$. By definition, $\tau = \sum_{i=1}^l m_i \omega_i \in \Lambda^+$ if and only if $m_i > 0$ are all integers. Also, $\sum_{i=1}^l \omega_i = \frac{1}{2} \sum \alpha$, where $\alpha \in \Phi^+$ is the set of all positive roots. All $L$-modules in this paper are finite dimensional.

Let $W$ denote the Weyl group of $L$. $W$ is generated by the simple reflections \(\{\sigma_1, \ldots, \sigma_l\}\) where $\sigma_i(x) = x - \langle x, \alpha_i \rangle \alpha_i$. For any $i$, $1 < i < l$, we define $W(i)$ to be the subgroup of $W$ generated by \(\{\sigma_j | j \neq i, 1 < j < l\}\). Note that each element of $W(i)$ fixes $\omega_i$.

In all of what follows, the set of weights of the irreducible $L$-module $V_{\lambda}$ will be denoted by $\Pi$.

We shall prove

**Theorem 1.** Let $V_{\lambda}$ be the irreducible $L$-module of highest weight $\lambda$. Let $\tau = \sum_{i=1}^l m_i \omega_i \in \Lambda^+$ and $V_{\lambda} \otimes V_{\tau} = \sum_{\gamma \in \Lambda^+, r_{\gamma} V_{\gamma}}$. Then for each $i$, $1 < i < l$, there is a positive integer $n_i$ depending only on $\lambda$, such that if $m_i > n_i$, then $V_{\lambda} \otimes V_{\tau + \omega_i} = \sum_{\gamma \in \Lambda^+, r_{\gamma} V_{\gamma + \omega_i}}$.

We shall give explicit values for the $n_i$ in terms of $\lambda$.

Theorem 1 should be compared with a result of Kostant [4]. He puts a much stronger requirement on $\tau$, namely that $\mu + \tau$ is dominant for every $\mu \in \Pi$. Under this condition, one can read off the decomposition of $V_{\lambda} \otimes V_{\tau}$ from the weight-space decomposition of $V_{\lambda}$: $V_{\lambda} \otimes V_{\tau} = \sum_{\mu \in \Pi} \text{Mult}_{\lambda}(\mu) V_{\mu + \tau}$.

The conclusion of Theorem 1 clearly follows for such $\tau$. However, Kostant's condition is satisfied only by dominant weights $\tau$ well into the interior of the fundamental chamber, and gives no information about infinitely many weights on or near the chamber walls. Theorem 1, on the other hand,
expresses a condition of uniformity along lines in the decomposition of the tensor product \( V \otimes V \), whenever \( \tau \) is outside a specified finite region.

If we let \( S(i) = \bigcup_{\sigma \in W(i)} \sigma(\Lambda^+) \), then \( n_i \) may be chosen as the least positive integer such that for each \( \mu \in \Pi \) we have \( \mu + n_i \omega_i \in S(i) \).

**Corollary 1.** Let \( V \) be fixed. Let \( (n_1, \ldots, n_l) \) be the \( l \)-tuple of positive integers which can be found by the above theorem. If we know the decompositions into irreducible \( L \)-modules of the finite set of tensor products \( \{ V \otimes V | \tau = \sum_{j=1}^l m_j \omega_j \text{ and } m_j < n_j \text{ for all } j, 1 < j < l \} \), then we know the decomposition of the tensor product of \( V \) with any irreducible \( L \)-module.

Let \( i, 1 < i < l \), be fixed throughout the following and let \( S = S(i) \).

**Lemma 1.** \( \bigcup_{\sigma \in W(i)} \sigma(\Lambda^+) = \{ x \in \Lambda^+ | (x, \sigma a) > 0, \forall \sigma \in W(i) \} \).

**Proof.** Let \( S = \bigcup_{\sigma \in W(i)} \sigma(\Lambda^+) \) and \( S' = \{ x \in \Lambda^+ | (x, \sigma a) > 0, \forall \sigma \in W(i) \} \). If \( x \in \Lambda^+ \) then \( (x, a_j) > 0 \) for \( 1 < j < l \), so for any \( \sigma \in \Phi^+ \), \( (x, a) > 0 \). For any \( \sigma \in W(i) \), \( \sigma a \in \Phi^+ \) because it is certainly a root and has +1 as its \( a \)-coefficient, so all coefficients are nonnegative. It follows that \( (x, \sigma a) > 0 \); that is, \( x \in S' \). Thus, \( \Lambda^+ \subseteq S' \). For any \( x \in S' \) and any \( \sigma, \sigma' \in W(i) \), \( (x, \sigma \sigma') = (x, \sigma^{-1} \sigma') > 0 \) since \( \sigma^{-1} \sigma' \in W(i) \). This means that if \( x \in S' \) then \( \sigma x \in S' \) for any \( \sigma \in W(i) \). From \( \Lambda^+ \subseteq S' \) we then get \( S \subseteq S' \).

Suppose there is an \( x \in S' \), \( x \notin S \). In the finite set \( \{ \sigma x \sigma | \sigma \in W(i) \} \) let \( \sigma x \) be chosen such that \( (\sigma x, \delta) \) is maximal. Since \( x \notin S \), \( \sigma x \notin \Lambda^+ \) and there is a \( j, 1 < j < l \), such that \( (\sigma x, a_j) < 0 \). If \( j \neq i \) then \( \sigma_j \in W(i) \) and \( \sigma \sigma_j \in W(i) \). But \( (\sigma_j \sigma x, \delta) = (\sigma x, \delta - a_j) = (\sigma x, \delta) - (\sigma x, a_j) > (\sigma x, \delta) \), contradicting the choice of \( \sigma x \). So \( j = i \) and \( (\sigma x, \sigma^{-1} a_i) = (\sigma x, a_i) < 0 \). But since \( \sigma^{-1} \in W(i) \), this contradicts \( x \in S' \), giving \( S = S' \).

**Lemma 2.** There is an integer \( n_i > 0 \) such that for any \( \mu \in \Pi \), \( \mu + n_i \omega_i \in S \). The least such \( n_i \) is \( \text{Max} \{ (\mu, a_i) | \mu \in \Pi \} \).

**Proof.** For any \( \sigma \in W(i) \), \( (n_i \omega_i, \sigma a_i) = n_i (\sigma^{-1} \omega_i, a_i) = n_i (\omega_i, a_i) = n_i (a_i, a_i)/2 \). The conditions on \( n_i \) equivalent to \( \mu + n_i \omega_i \in S \) for all \( \mu \in \Pi \) are \( 0 < (\mu + n_i \omega_i, \sigma a_i) = (\mu, \sigma a_i) + (n_i \omega_i, \sigma a_i) = (\mu, \sigma a_i) + n_i (a_i, a_i)/2 \) for all \( \mu \in \Pi \) and all \( \sigma \in W(i) \). That is, \( n_i > -2(\mu, \sigma a_i)/(a_i, a_i) = -\langle \mu, \sigma a_i \rangle = -\langle \sigma^{-1} \mu, a_i \rangle = \langle \sigma^{-1} \mu, a_i \rangle = \langle \sigma^{-1} a_i, a_i \rangle = \langle a_i, a_i \rangle \). Since \( \Pi \) is invariant under \( W \), \( \{ \sigma a^{-1} \mu | \mu \in \Pi \), \( a \in W(i) \} = \Pi \). We now have the finite number of conditions \( n_i > \langle a_i, a_i \rangle \) for all \( \mu \in \Pi \) which has least solution

\[ n_i = \text{Max} \{ \langle a_i, a_i \rangle | \mu \in \Pi \} > 0. \]

**Lemma 3.** For any \( \gamma_1, \gamma_2 \in S \), \( \gamma_1 + \gamma_2 \in S \).

**Proof.** Clear from Lemma 1.

**Proof of Theorem 1.** If we use the notation

\[ T_\lambda = \sum_{\sigma \in W} \text{sgn}(\sigma) \exp(\sigma(\lambda + \delta)), \]
then the Weyl character formula says $X_\lambda \cdot T_0 = T_\lambda$, where $X_\lambda$ is the character of a representation $V_\lambda$ of highest weight $\lambda$. Then the character of $V_\lambda \otimes V_\tau$ is $X_\lambda \cdot X_\tau$. After some elementary manipulations, one sees that

$$X_\lambda \cdot X_\tau \cdot T_0 = \sum_{\mu \in \Pi} \text{Mult}_\lambda(\mu) \cdot T_{\mu+\tau}.$$ 

Replacing $\tau$ by $\tau + \omega_\lambda$, we also have

$$X_\lambda \cdot X_{\tau+\omega_\lambda} \cdot T_0 = \sum_{\mu \in \Pi} \text{Mult}_\lambda(\mu) \cdot T_{\mu+\tau+\omega_\lambda}.$$ 

By Lemma 2, $\mu + n_1 \omega_\lambda \in S$ for all $\mu \in \Pi$. If $\tau = \sum_{j=1}^l m_j \omega_j$ satisfies $m_j > n_j$, then $\tau - n_1 \omega_1 \in \Lambda^+ \subseteq S$. By Lemma 3, $\mu + \tau = (\mu + n_1 \omega_1) + (\tau - n_1 \omega_1) \in S$. Of course, $\omega_\lambda, \delta \in \Lambda^+ \subseteq S$ and so $\mu + \delta + \tau \in S$ as well as $\mu + \delta + \tau + \omega_\lambda \in S$. This means that both $\mu + \delta + \tau$ and $\mu + \delta + \tau + \omega_\lambda$ are conjugate by elements of $W(i)$ to dominant weights. In fact, they are conjugate by the same element because if $\sigma_\mu(\mu + \delta + \tau) \in \Lambda^+$ for $\mu \in W(i)$ then $\sigma_\mu(\mu + \delta + \tau + \omega_\lambda) = \sigma_\mu(\mu + \delta + \tau + \omega_\lambda) \in \Lambda^+$. Thus

$$T_{\mu+\tau} = \sum_{\sigma \in W} \text{sgn}(\sigma) \exp(\sigma(\mu + \tau + \delta))$$

and

$$T_{\mu+\tau+\omega_\lambda} = \sum_{\sigma \in W} \text{sgn}(\sigma) \exp(\sigma(\mu + \tau + \omega_\lambda + \delta)).$$

This means that $T_{\mu+\tau}/T_0 = \text{sgn}(\sigma_\mu) \cdot X_{\sigma_\mu(\mu+\tau+\delta)-\delta}$ and $T_{\mu+\tau+\omega_\lambda}/T_0 = \text{sgn}(\sigma_\mu) \cdot X_{\sigma_\mu(\mu+\delta+\tau)-\delta+\omega_\lambda}$. Then

$$X_\lambda \cdot X_\tau = \sum_{\mu \in \Pi} \text{Mult}_\lambda(\mu) \cdot \text{sgn}(\sigma_\mu) \cdot X_{\sigma_\mu(\mu+\tau+\delta)-\delta}$$

and

$$X_\lambda \cdot X_{\tau+\omega_\lambda} = \sum_{\mu \in \Pi} \text{Mult}_\lambda(\mu) \cdot \text{sgn}(\sigma_\mu) \cdot X_{\sigma_\mu(\mu+\tau+\delta)-\delta+\omega_\lambda}.$$ 

Grouping equivalent terms together, if $X_\lambda \cdot X_\tau = \sum_{\gamma \in \Lambda^+} \tau_\gamma X_\gamma$ then the above shows that $X_\lambda \cdot X_{\tau+\omega_\lambda} = \sum_{\gamma \in \Lambda^+} \tau_\gamma X_{\gamma+\omega_\lambda}$ as claimed by the theorem.

It should be noted that the author originally based his proof on a formula of Klimyk [3], whose geometric nature was essential to the discovery of this result. The present proof, based on the closely related and well-known Weyl character formula, was suggested by the referee.

Note that if $\tau = \sum_{j=1}^l m_j \omega_j$ satisfies $m_j > n_j$ for all $j$, $1 < j < l$, then by Lemma 2, for each $\mu \in \Pi$ and each $j$, $\mu + n_j \omega_j \in S(j)$ and $\tau - n_j \omega_j \in \Lambda^+ \subseteq S(j)$. Then $\mu + \tau \in S(j)$, which means $\mu + \tau \in \cap_{1 \leq j \leq l} S(j)$ for all $\mu \in \Pi$.

For each $j$, $\Lambda^+ \subseteq S(j)$, so $\Lambda^+ \subseteq \cap_{1 \leq j \leq l} S(j)$. For each $j$, $S(j) \subseteq \{ x \in \}$.
\( \Lambda (x, \alpha) > 0 \). This means that \( \cap_{1 < j < l} S(j) \subseteq \{ x \in \Lambda | (x, \alpha_j) > 0 \} \) for all \( 1 < j < l \) = \( \Lambda^+ \). We now have \( \cap_{1 < j < l} S(j) = \Lambda^+ \).

The above says that for each \( \mu \in \Pi, \mu + \tau \in \Lambda^+ \), which is the condition required by Kostant's theorem. Although Kostant used a theorem of Brauer [1] in his proof, Weyl's formula also gives the result as follows. If \( \mu + \tau \in \Lambda^+ \) then \( \mu + \delta + \tau \) is strictly dominant, so \( \sigma_\mu = 1 \) and \( X_{\delta_\mu (\mu + \delta + \tau) - \delta} = X_{\mu + \tau} \). Weyl's formula then says \( X^{(\lambda)} \cdot X^\tau = \sum_{\mu \in \Pi} \text{Mult}_\lambda (\mu) \cdot X^{\mu + \tau} \) which is now a direct sum.

In fact, if \( \tau_0 = \sum_{j=1}^l \eta_j \omega_j \) then no “smaller” dominant weight \( \tau' \) satisfies \( \mu + \tau' \in \Lambda^+ \) for all \( \mu \in \Pi \). If we let \( \tau' = \sum_{j=1}^l q_j \omega_j \) with some \( q_p < \eta_p \), then the condition \( \mu + \tau' \in \Lambda^+ \) for all \( \mu \in \Pi \) means \( \langle \mu + \tau', \alpha_i \rangle > 0 \) for \( 1 < i < l \). That is, \( \langle \mu, \alpha_i \rangle + \sum_{j=1}^l q_j \langle \omega_j, \alpha_i \rangle = \langle \mu, \alpha_i \rangle + q_i > 0 \). Or \( q_i > -\langle \mu, \alpha_i \rangle = \langle \mu, \sigma_i \rangle = \langle \sigma_i \mu, \alpha_i \rangle \) for all \( \mu \in \Pi \) and \( 1 < i < l \). As in Lemma 2, \( \{ \sigma_i \mu | \mu \in \Pi \} = \Pi \), so \( q_i > \langle \mu, \alpha_i \rangle \). But from Lemma 2, \( \eta_p = \text{Max} \{ \langle \mu, \alpha_p \rangle | \mu \in \Pi \} \). Therefore \( q_p > \eta_p \), which contradicts \( q_p < \eta_p \). This shows that the Kostant region of uniform decomposition is precisely \( \{ \tau_0 + \gamma | \gamma \in \Lambda^+ \} \).

**Lemma 4.** \( n_1 = \text{Max} \{ \langle \mu, \sigma_1 \alpha_1 \rangle | \mu \in \Pi \cap \Lambda^+ \}, \sigma \in W \} \).

**Proof.** From Lemma 2, \( n_i = \text{Max} \{ \langle \mu, \alpha_i \rangle | \mu \in \Pi \} \). Every \( \mu \in \Pi \) is conjugate to some dominant weight in \( \Pi \), giving the lemma.

We can now give the sharper result.

**Lemma 5.** \( n_i = \langle \lambda, \theta_i \alpha_i \rangle \) where \( \theta_i \in W \) is such that \( \theta_i \alpha_i \) is the highest root conjugate to \( \alpha_i \). Thus, for \( L \) simple, if \( \alpha_i \) is a short root, \( \theta_i \alpha_i \) is the highest short root, and if \( \alpha_i \) is a long root, \( \theta_i \alpha_i \) is the highest long root.

**Proof.** Fix \( \mu \in \Pi \cap \Lambda^+ \). Then for any \( \sigma \in W \), \( \langle \mu, \sigma \alpha_i \rangle = 2(\mu, \sigma \alpha_i)/(\alpha_i, \alpha_i) \) and \( \langle \mu, \theta_i \alpha_i \rangle - \langle \mu, \sigma \alpha_i \rangle = 2(\mu, \theta_i \alpha_i - \sigma \alpha_i)/(\alpha_i, \alpha_i) > 0 \) since \( \mu \) is dominant and \( \theta_i \alpha_i - \sigma \alpha_i \) is a nonnegative sum of positive roots. So \( n_i = \text{Max} \{ \langle \mu, \theta_i \alpha_i \rangle | \mu \in \Pi \cap \Lambda^+ \} \). It is a well-known fact that \( \theta_i \alpha_i \) is dominant, and since \( \lambda - \mu \) is a nonnegative sum of positive roots, we have \( \langle \lambda, \theta_i \alpha_i \rangle - \langle \mu, \theta_i \alpha_i \rangle = \langle \lambda - \mu, \theta_i \alpha_i \rangle > 0 \). This says the maximum is attained at \( \langle \lambda, \theta_i \alpha_i \rangle \).

This precise characterization of \( n_i \) allows us to calculate the \( l \)-tuple, \((n_1, \ldots, n_l)\), for each type of algebra in terms of \( \lambda = \sum_{i=1}^l m_i \omega_i \). I have labeled the Dynkin diagrams as in [2]. The results are:

- **A**: \( n_i = m_i + m_2 + \cdots + m_l \) for \( 1 \leq i \leq l \),
- **B**: \( n_i = m_i + 2m_2 + \cdots + 2m_{l-1} + m_l \) for \( 1 \leq i \leq l - 1 \),
- **C**: \( n_i = m_i + 2m_2 + \cdots + 2m_{l-1} + 2m_l \) for \( 1 \leq i \leq l - 1 \),
- **D**: \( n_i = m_i + 2m_2 + \cdots + 2m_{l-2} + m_{l-1} + m_l \) for \( 1 \leq i \leq l \),
- **E**: \( n_i = m_i + 2m_2 + 2m_3 + 3m_4 + 4m_5 + 5m_6 \) for \( 1 \leq i \leq 6 \),
- **F**: \( n_i = 2m_1 + 2m_2 + 3m_3 + 4m_4 + 5m_5 + 6m_6 + m_7 \) for \( 1 \leq i \leq 7 \),
$E_8$: $n_i = 2m_1 + 3m_2 + 4m_3 + 6m_4 + 5m_5 + 4m_6 + 3m_7 + 2m_8$

for $1 < i < 8$,

$F_4$: $n_1 = n_2 = 2m_1 + 3m_2 + 2m_3 + m_4,$

$n_3 = n_4 = 2m_1 + m_2 + 3m_3 + 2m_4,$

$G_2$: $n_1 = 2m_1 + 3m_2,$

$n_2 = m_1 + 2m_2.$

For $L$ semisimple, these formulas are applied to each simple component separately. If $\alpha_i$ is in a certain component of the Dynkin diagram of $L$, then the highest root conjugate to $\alpha_i$ involves only the roots in that component. So $n_i$ is calculated according to the type of that component and is given by one of the above formulas involving only those $m_j$ such that $\alpha_j$ is in that component.

REFERENCES


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