A NOTE ON NONUNITARY PRINCIPAL SERIES REPRESENTATIONS

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Abstract. It is proven in a rather elementary way that any nonunitary principal series representation of a semisimple Lie group $G$ is of finite length, having as a trivial consequence that the set of infinitesimal equivalence classes of quasisimple irreducible representations of $G$ with a given infinitesimal character is finite.

1. Introduction. The aim of this paper is to prove a theorem asserting that any nonunitary principal series representation of a semisimple Lie group $G$ is of finite length. This theorem has been known already for a long time. However, its proof was based on the fact that the set of infinitesimal equivalence classes of irreducible representations with a given infinitesimal character is finite. This latter fact was proven using a very deep result of Harish-Chandra— that all invariant eigendistributions on $G$ are representable by locally integrable functions on $G$.

In [4] the above-mentioned theorem was proven in a more elementary way in the case that $G$ is linear, but the proof does not generalize to nonlinear groups. The authors use rather careful investigation of finite dimensional representations of $G$, and Kostant’s (or, more elementary, Helgason’s) result on cyclicity of a $K$-fixed vector in some spherical nonunitary principal series representations.

Our method to prove the theorem (for general $G$) is similar but seems even simpler, because it does not use such fine results on finite dimensional representations. Also, instead of Helgason’s result on cyclicity, we use a more elementary theorem of Milicic.

2. Notation. For any Lie group $H$ we denote by $\hat{H}$ the set of equivalence classes of irreducible finite dimensional continuous representations of $H$ on complex vector spaces. We shall usually not distinguish between a representation and its equivalence class. If $\sigma$ is a finite dimensional continuous representation of $H$, $\bar{\sigma}$ will stand for the contragredient representation.

If $S$ is a subset of a complex vector space, span $S$ will denote the subspace spanned by $S$. The subscript “$c$” will denote the complexification of a real vector space.

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3. The main result. Let $G$ be a connected semisimple Lie group with Lie algebra $\mathfrak{g}$, $G = KAN$ an Iwasawa decomposition of $G$ and $\mathfrak{g} = \mathfrak{t} + \mathfrak{a} + \mathfrak{n}$ the corresponding decomposition of $\mathfrak{g}$. Let $M$ be the centralizer of $A$ in $K$. Then $P = MAN$ is a closed subgroup of $G$—a minimal parabolic subgroup.

Let $\mathfrak{g}$ be the universal enveloping algebra of $\mathfrak{g}_c$.

Let $\sigma$ be a continuous representation of $P$ on a finite dimensional complex space $V$. Denote by $V^\sigma$ the space of all functions $f \in C^\infty(G, V)$ with the following properties:

(i) $f(px) = \sigma(p)f(x), p \in P, x \in G$;

(ii) $\dim \text{span}\{f^k; k \in \mathbb{K}\} < \infty$, where $f^k(x) = f(xk), x \in G$. For $X \in \mathfrak{g}$ and $f \in V^\sigma$ let $\pi^\sigma(X)f: G \rightarrow V$ be defined by

$$
(\pi^\sigma(X)f)(x) = \frac{d}{dt}|_{t=0}f(x \exp tx), \quad x \in G.
$$

Then $\pi^\sigma(X)f \in V^\sigma$ and $\pi^\sigma: X \mapsto \pi^\sigma(X)$ is a representation of $\mathfrak{g}$ on $V^\sigma$. $\pi^\sigma$ extends to $\mathfrak{g}$, hence $V^\sigma$ becomes a $\mathfrak{g}$-module. It is also a $K$-module under the right translation of functions.

If $\tau$ is a finite dimensional representation of $M$ on $V$ and $\lambda \in \mathfrak{a}_c^*$, we define the representation $(\tau, \lambda)$ of $P$ on $V$ by

$$(\tau, \lambda)(man) = e^{\lambda(\log a)}\tau(m), \quad m \in M, a \in A, n \in N.$$ 

We have then $\hat{P} = \{(\tau, \lambda); \tau \in \hat{M}, \lambda \in \mathfrak{a}_c^*\}$. $V^{(\tau, \lambda)}, \tau \in \hat{M}, \lambda \in \mathfrak{a}_c^*$, are called nonunitary principal series $\mathfrak{g}$-modules.

**Theorem.** Every nonunitary principal series $\mathfrak{g}$-module is of finite length.

**Proof.** Let $\Sigma$ be the restricted root system of $\mathfrak{g}$ in $\mathfrak{a}_c^* \subset \mathfrak{a}_c^*$. The chosen Iwasawa decomposition fixes the set $\Sigma^+$ of positive restricted roots. For any $\alpha \in \Sigma$ let $\mathfrak{g}^\alpha$ be the corresponding root subspace of $\mathfrak{g}$ and let $H_\alpha$ be the unique element of $a \cap [\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}]$ such that $\alpha(H_{\alpha}) = 2$. Furthermore, set

$$
\rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} (\dim \mathfrak{g}^\alpha) \cdot \alpha.
$$

For $\gamma \in \hat{G}$ realized on $W$ let $\lambda_{\gamma}$ denote the lowest restricted weight (with respect to $\Sigma^+$) of the $\mathfrak{g}$-module $W$. Let $W^-$ be the corresponding restricted weight subspace of $W$ and let $W^+$ be the sum of all the other restricted weight subspaces. Then $W^-$ is invariant and irreducible under $M$ and we denote by $\tau_\gamma$ the corresponding element of $\hat{M}$. $W^+$ is a $P$-submodule of $W$ and the representation of $P$ on $W/W^+$ is $(\tau_\gamma, \lambda_\gamma)$. Hence, $(\tau_\gamma, \lambda_\gamma)$ is a quotient of the representation $\gamma|P$. $\lambda_\gamma$ assumes real values on $a$ and $\lambda_\gamma(H_\alpha) < 0$ for any $\alpha \in \Sigma^+$. Furthermore, for any $\mu \in \mathfrak{a}^*$ there exists $\gamma \in \hat{G}$ such that $\lambda_\gamma(H_\alpha) < \mu(H_\alpha)$ for every $\alpha \in \Sigma^+$.

Let $\omega \in \hat{M}$ and $\nu \in \mathfrak{a}_c^*$. We want to prove first that the $\mathfrak{g}$-module $V^{(\omega, \nu)}$ is finitely generated.

Choose $\gamma \in \hat{G}$ so that $\lambda_\gamma(H_\alpha) < \Re \nu(H_\alpha) - \rho(H_\alpha)$ for any $\alpha \in \Sigma^+$. Set $\lambda = \nu - \lambda_\gamma$. Let $\sigma \in \hat{M}$ be contragredient to some irreducible constituent of $\tau_\gamma \otimes \tilde{\omega}$. Then $\sigma \otimes \tau_\gamma \otimes \tilde{\omega}$ fixes a nonzero vector, hence $\omega$ is a direct summand.
in $\sigma \otimes \tau_\gamma = (\tau_\gamma, \lambda_\gamma)$ being a quotient of $\gamma|P$, it follows that $(\omega, \nu) = (\omega, \lambda + \lambda_\gamma)$ is a quotient of $(\sigma, \lambda) \otimes (\gamma|P)$. Let $W$ be the space of $\gamma$. By Proposition (10.6) in [2], there exists a surjective $\mathfrak{g}$-morphism of $V^{(\sigma, \lambda)} \otimes W$ onto $V^{(\omega, \nu)}$. If we prove that $V^{(\sigma, \lambda)}$ is finitely generated, Proposition 3.3 in [3] will imply that $V^{(\sigma, \lambda)} \otimes W$, hence also $V^{(\omega, \nu)}$, is finitely generated.

Now, $\text{Re}(2\rho - \lambda)(H_\alpha) = \text{Re}(2\rho - \nu + \lambda_\gamma)(H_\alpha) < \rho(H_\alpha)$ for every $\alpha \in \Sigma^+$. By Theorem 3.1 in [6], $V^{(\sigma, 2\rho - \lambda)}$ contains the smallest $\mathfrak{g}$-submodule different from $\{0\}$. There exists a canonical $\mathfrak{g}$-invariant (and $K$-invariant) nondegenerate bilinear form on $V^{(\sigma, 2\rho - \lambda)} \times V^{(\sigma, \lambda)}$ (see e.g. Lemma 5.1 in [4]), hence $V^{(\sigma, \lambda)}$ contains a proper $\mathfrak{g}$-submodule $U$ which contains any other proper $\mathfrak{g}$-submodule of $V^{(\sigma, \lambda)}$. But then every vector in $V^{(\sigma, \lambda)}$ not contained in $U$ generates the $\mathfrak{g}$-module $V^{(\sigma, \lambda)}$. Especially, $V^{(\sigma, \lambda)}$ is finitely generated.

Thus, we have proved that every nonunitary principal series module is finitely generated. $\mathfrak{g}$ being Noetherian, all of these modules are Noetherian. By the above mentioned canonical pairing on $V^{(\sigma, 2\rho - \lambda)} \times V^{(\omega, \nu)}$ all of these modules are also Artinian. Hence, they are of finite length. Q.E.D.


COROLLARY. Suppose that $G$ has finite center. Let $\mathfrak{z}$ be the center of $\mathfrak{g}$ and $\chi \in \text{Hom}(\mathfrak{z}, \mathbb{C})$. There exist only finitely many equivalence classes of irreducible $(\mathfrak{g}, K)$-modules with infinitesimal character $\chi$.

PROOF. By Casselman’s theorem [1], [5], [6] every irreducible $(\mathfrak{g}, K)$-module is equivalent to a submodule of some $V^{(\sigma, \lambda)}$, $\sigma \in \tilde{M}$, $\lambda \in \mathfrak{a}_\sigma^*$. The number of pairs $(\sigma, \lambda) \in \tilde{M} \times \mathfrak{a}_\sigma^*$, such that $V^{(\sigma, \lambda)}$ has infinitesimal character $\chi$, being obviously finite, the Corollary follows immediately from the Theorem. Q.E.D.

REMARK 1. It is easy to generalize the Theorem to the case of a reductive Lie group $G$ such that $ZG^0$ is of finite index in $G$, where $Z$ is the center and $G^0$ is the identity component of $G$. In this case nonunitary principal series modules are $(\mathfrak{g}, K)$-modules induced by finite dimensional irreducible representations of $P$, where $P$ is the normalizer of a minimal parabolic subalgebra of $[\mathfrak{g}, \mathfrak{g}]$ and $K$ is the centralizer in $G$ of a Cartan involution on $[\mathfrak{g}, \mathfrak{g}]$.

REMARK 2. Let $G$ be a group in the Harish-Chandra class, $Q$ any parabolic subgroup of $G$, $N$ the unipotent radical of $Q$, $L$ a Levi factor of $Q$. Let $\sigma$ be an irreducible quasisimple representation of $L$ extended to the representation of $P$ (trivial on $N$). By Casselman’s theorem [1] $\sigma$ is (infinitesimally) equivalent to a subrepresentation of some nonunitary principal series representation of $L$. An induction in stages argument shows that the induced representation $\pi^\sigma$ of $G$ is equivalent to a subrepresentation of some nonunitary principal series representation of $G$. Especially, $\pi^\sigma$ is of finite length.

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REFERENCES


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