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# $K(\mathbf{Z}/2)$ AS A THOM SPECTRUM

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ABSTRACT. Dyer-Lashof operations are used to give a simple proof that $K(\mathbf{Z}/2)$ is a Thom spectrum.

The purpose of this note is to give a simple proof of Mahowald's striking observation that the Eilenberg-Mac Lane spectrum $K(\mathbf{Z}/2)$ is a Thom spectrum. Our proof follows closely a proposed (but incorrect) proof by Madsen and Milgram [2]. Although Mahowald has given another proof [3], the idea of [2] is so appealing that it seems useful to record a corrected version.

Let $\eta\colon S^1 \to BO$ represent the generator of $\pi_1 BO = \mathbf{Z}/2$. Since $BO$ is a double loop space there is an induced map

$$\gamma\colon \Omega^2 S^3 = \Omega^2\Sigma^2(S^1) \xrightarrow{\Omega^2\Sigma^2\eta} \Omega^2\Sigma^2 BO \to BO.$$

Let $M(\gamma)$ denote the Thom spectrum associated with $\gamma$ (localized at 2).

THEOREM (MAHOWALD). $M(\gamma) \cong K(\mathbf{Z}/2)$.

PROOF. We recall that the Dyer-Lashof operation $Q_1$ is defined for double loop spaces and that $H_*\Omega^2 S^3 = \mathbf{Z}/2[x_1, x_2, \ldots, x_k, \ldots]$ where $x_k = Q_1 \cdots Q_1 \iota_1$ the $(k-1)$ fold iterate of $Q_1$ applied to the fundamental class $\iota_1$ ($\mathbf{Z}/2$ coefficients are used throughout). Also $H_*K(\mathbf{Z}/2) = \mathbf{Z}/2[\xi_1, \xi_2, \ldots, \xi_k, \ldots] = A_*$ the dual of the mod 2 Steenrod algebra $A$. Let $\alpha\colon A \to H^*MO$ denote evaluation on the Thom class $U$, i.e. $\alpha(a) = aU$, $\alpha_*$ is an algebra morphism in homology. Finally, let $\Gamma\colon M(\gamma) \to MO$ denote the map of Thom spectra induced by $\gamma$; $\Gamma_*$ is also an algebra morphism in homology.

Since $\deg x_k = 2^k - 1 = \deg \xi_k$, $H_*\Omega^2 S^3 \approx H_*M(\gamma)$ and $A_*$ have the same rank in each dimension and so the theorem follows from

LEMMA.

$$H_*M(\gamma) \xrightarrow{\Gamma_*} H_*MO \xrightarrow{\alpha_*} A_*$$

*is surjective.*

PROOF. Consider the commutative diagram

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$$\begin{array}{ccc} H_*M(\gamma) & \xrightarrow{\Gamma_*} & H_*MO \\ \Phi_*\uparrow & & \Phi_*\uparrow \\ H_*\Omega^2S^3 & \xrightarrow{\gamma_*} & H_*BO \end{array}$$

where $\Phi_*$ is the Thom isomorphism. Now $\gamma_*(x_k) = Q_1 \cdots Q_1\gamma(\iota_1) = Q_1 \cdots Q_1 p_1 = p_k$ by Kochman [**1**, Corollary 35], where $p_k \in H_{2^k-1}BO$ is the nonzero primitive element. Thus $\Phi_*(p_k) \in \text{Image}\,\Gamma_*$ and to complete the proof it suffices to show $\alpha_*\Phi_*(p_k) = \xi_k +$ decomposables. Let $q_k$ be dual to $\xi_k$; $q_k$ is primitive and since $\Phi^*$ is a coalgebra map $\Phi^* q_k U$ is also primitive, hence $\Phi^* q_k U = \mathcal{P}_k$ the nonzero primitive of $H^{2^k-1}BO$. Thus

$$\langle q_k, \alpha_*\Phi_* p_k\rangle = \langle q_k U, \Phi_* p_k\rangle = \langle \Phi^* q_k U, p_k\rangle = \langle \mathcal{P}_k, p_k\rangle = 1$$

since we are in an odd dimension. Q.E.D.

## References

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