AVOIDING SELF-REFERENTIAL STATEMENTS

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Abstract. Recursion-theoretic proofs of metamathematical results tend to rely on a pair of effectively inseparable r.e. sets and its properties. We establish a special property for a small configuration of such pairs and derive from it some metamathematical results not previously accessible to recursion-theoretic techniques.

0. Introduction. The applications of the dual completeness of a pair of effectively inseparable r.e. sets to metamathematical questions are manifold. Since Shepherdson 1960, however, more powerful results have been obtainable by diagonalization within a given theory. In this note, we prove a generalization of Smullyan's dual completeness result (cf. Rogers 1967, Exercise 11.29) and list some metamathematical corollaries not previously obtainable recursion-theoretically.

We let \([e]\) denote the partial recursive function with index \(e\), and \(W_e\) the r.e. set with index \(e\). \(Texv\) is Kleene's \(T\)-predicate and, for any assertions, \(3vRv\), \(3vSv\), with \(R, S\) recursive, we write
\[3vRv \leq 3vSv: 3v[RV \land \forall v' < v \exists v'],\]
\[3vRv < 3vSv: 3v[RV \land \exists v' < v \exists v'].\]

A disjunction \(3vTv \lor 3vUv\) in one of these contexts is assumed rewritten \(3v(Tv \lor Uv)\). For r.e. sets \(X, Y\), we define
\[X \leq Y: \{x: x \in X \leq x \in Y\}, \quad X < Y: \{x: x \in X < x \in Y\},\]
where \(x \in X, x \in Y\) abbreviate \(3vTexv\) for appropriate \(e\). Note that \(X \leq Y\) and \(Y < X\) are simply the sets obtained by applying the Reduction Theorem to \(X, Y\). (This notation is due to Dave Guaspari.)

1. A double dual completeness theorem. The main result of this note is the following

Theorem. Let \((A, C), (B, D)\) be pairs of effectively inseparable r.e. sets with \(A \subseteq B, C \subseteq D\). There is a recursive function \(f\) such that, for all \(x,\)
\[x \in A \iff fx \in A \iff fx \in B;\]
\[x \in C \iff fx \in C \iff fx \in D.\]

In words, the conclusion of the theorem simply states that the pair \((A, C)\) is uniformly many-one reducible to both pairs \((A, C)\) and \((B, D)\).

Proof. The proof is simple but devious. By Smullyan's dual completeness...
result, there is a recursive function $g$ such that, for all $i, j$, the function 
$[g(i, j)]$ reduces the pair $(W_i \leq W_j, W_j < W_i)$ to $(A, C)$. Apply Smullyan's
Double Recursion Theorem (Rogers 1967, Theorem 11.10) to obtain indices
$a, c$ such that, for $f = [g(a, c)]$ and all $x$,

\[
\begin{align*}
  x \in W_a &\iff [fx \in D \lor x \in A, \leq \ldots, fx \in B \lor x \in C], \\
  x \in W_c &\iff [fx \in B \lor x \in C, < \ldots, fx \in D \lor x \in A].
\end{align*}
\]

Obviously, $W_a$ and $W_c$ are disjoint.

Claim 1. $W_a = A \leq C = A; W_c = C < A = C$.

To see this, observe

\[
\begin{align*}
  x \in W_a &\Rightarrow x \in W_a - W_c \\
  &\Rightarrow fx \in A \subseteq B \land fx \notin D, \text{ since } A \cap D = \emptyset \\
  &\Rightarrow x \in A, \leq \ldots, fx \in B \lor x \in C, \text{ by definition of } W_a \\
  &\Rightarrow x \in A.
\end{align*}
\]

Similarly, $x \in W_c \Rightarrow x \in C$. But also,

\[
\begin{align*}
  x \in A &\Rightarrow x \in W_a \lor x \in W_c \Rightarrow x \in W_a,
\end{align*}
\]

since $x \in W_c$ yields $x \in C$ which is disjoint from $A$. Similarly $x \in C \Rightarrow x \in W_c$.

Claim 2. For all $x$,

\[
\begin{align*}
  x \in A &\iff fx \in A, \quad x \in C \iff fx \in C.
\end{align*}
\]

This is trivial since $f = [g(a, c)]$ and $(A, C) = (W_a, W_c) = (W_a \leq W_c,$

\[W_c < W_a).\]

Claim 3. For all $x$,

\[
\begin{align*}
  x \in A &\iff fx \in B, \quad x \in C \iff fx \in D.
\end{align*}
\]

The left-to-right implications follow from Claim 2. For the other direction,
assume first that $fx \in B$. A glance at the definition of $W_a, W_c$ reveals that
$x \in W_a$ or $x \in W_c$. The latter yields $fx \in D$, contrary to assumption. Thus
$x \in W_a = A$. Similarly one shows $fx \in D$ implies $x \in C$. Q.E.D.

Obviously we can compose a reduction of $(X, Y)$ to $(A, C)$ with $f$ to obtain
a simultaneous reduction of any pair of disjoint r.e. sets to $(A, C)$ and $(B, D)$.
A second corollary, noticed by J. R. Shoenfield, is this: For $A, B, C, D$ as in
the Theorem, any set $X$ interpolated between $A$ and $B$, $A \subseteq X \subseteq B$, has
degree at least $\Theta$. [N.B. Without $C$ and $D$, this need not hold: Creative sets
can have recursive interpolants.]

2. Some metamathematical applications. We give a few corollaries concerning
the metamathematics of r.e. systems of arithmetic (for definiteness: extensions of Robinson's \( \mathbb{Q} \)) that were previously obtainable only via self-
referential formulae (cf. Shepherdson 1960, Smoryński A).

Definitions. A formula $\varphi x_0 \cdots x_{n-1}$ semirepresents a relation $R \subseteq \omega^n$ in a
theory $\mathcal{T}$ iff, for all $x_0, \ldots, x_{n-1}$,

\[
\mathcal{T} \vdash \varphi \bar{x}_0 \cdots \bar{x}_{n-1} \iff Rx_0 \cdots x_{n-1}.
\]
\( \varphi \text{ dually semirepresents} \) a disjoint pair of relations, \( R, S \) iff \( \varphi, \varphi \neg \text{ semirepresents} R, S \), respectively. \( \varphi \text{ represents} R \) iff \( \varphi \text{ dually semirepresents} R \) and its complement. A formula \( \varphi v_0 \cdots v_n \text{ semirepresents (represents)} \) a partial (total) function \( f \) iff (i) \( \varphi \text{ semirepresents (represents)} \) the graph of \( f \), and (ii) \( \varphi \) satisfies a unicity condition, say,

\[
\mathcal{T} \vdash \varphi v_0 \cdots v_{n-1} v \land \varphi v_0 \cdots v_{n-1} v' \rightarrow v = v'.
\]

[This is stronger than necessary for most purposes.]

**Corollary 1.** Let \( \mathcal{T} \) be a consistent r.e. extension of \( \mathcal{R} \). For any disjoint pair, \( R, S \) of \( n \)-ary r.e. relations, there is a formula \( \varphi v_0 \cdots v_{n-1} \in \Sigma_1 \) which dually semirepresents \( R, S \) in \( \mathcal{T} \); and, moreover, \( \varphi v_0 \cdots v_{n-1} \) defines \( R \) in the set of natural numbers.

**Proof.** Obviously we can assume the Theorem proven for \( n \)-ary relations. Moreover, by Smullyan's Dual Completeness Theorem, we can assume \( R, S \) to be effectively inseparable. So let \( \psi_0, \psi_1 \) be \( \Sigma_1 \) definitions of \( R, S \) and let \( A = R, C = S, B = \{(x_0, \ldots, x_{n-1}) : \exists (\psi_0), \exists x_0 \cdots \exists x_{n-1}\}, \) and \( D = \{(x_0, \ldots, x_{n-1}) : \exists (\psi_0), \exists x_0 \cdots \exists x_{n-1}\}. \) Now simply define

\[
\exists v_0 \cdots v'_{n-1} [v_0 \cdots v_{n-1} \land (\psi_0 \leq (\psi_1)v_0 \cdots v'_{n-1}],
\]

where \( \chi \in \Sigma_1 \) represents the recursive function \( f \) of the Theorem. Q.E.D.

The correctness of the semirepresentation of \( R \) is the novel feature of this proof. While it comes free with Shepherdson's proof via self-referential formulae, the correctness has either been lacking in recursion-theoretic proofs of Corollary 1 (Ehrenfeucht and Feferman 1960, Putnam and Smullyan 1960), or has resulted in non-\( \Sigma_1 \) semirepresentations (Hájková and Hájek 1972).

**Corollary 2.** The dual semirepresentation \( \varphi \) for disjoint \( R, S \) can be chosen uniformly in an r.e. sequence, \( \mathcal{T}_0, \mathcal{T}_1, \ldots \), of consistent extensions of \( \mathcal{R} \).

The proof is as before: Let \( B_i, D_i \) be the sets of tuples provably in, respectively out of, \( R \preceq S \) in \( \mathcal{T}_i \); and let \( B = \bigcup_i B_i, D = \bigcup_i D_i \). Now simply define

\[\exists v_0 \cdots v'_{n-1} [v_0 \cdots v_{n-1} \land (\psi_0 \leq (\psi_1)v_0 \cdots v'_{n-1}], \]

where \( \chi \in \Sigma_1 \) represents the recursive function \( f \) of the Theorem. Q.E.D.

Again, this result was originally quite easily proven by means of formal diagonalization.

**Corollary 3.** Let \( f \) be partial recursive; \( \mathcal{T}_0, \mathcal{T}_1, \ldots \) an r.e. sequence of consistent extensions of \( \mathcal{R} \). There is a formula \( \varphi v_0 \cdots v_n \in \Sigma_1 \) which correctly uniformly semirepresents \( f \) in each \( \mathcal{T}_i \). Moreover, we can assume

\[
\mathcal{T}_i \vdash \neg \varphi \exists x_0 \cdots x_{n-1} \exists y \equiv \exists z \neq y (fx_0 \cdot \cdot \cdot x_{n-1} = z).
\]

Again the result is sharper than the original recursion-theoretic result (Ritchie and Young 1968/1969). We omit the proof.

As a final application we have

**Corollary 4.** Let \( \mathcal{T}_0 \subset \mathcal{T}_1 \) be consistent r.e. extensions of \( \mathcal{R} \) and let
$R_0 \subseteq R_1$ be n-ary r.e. relations. There is a formula $\varphi$ such that $\varphi$ semirepresents $R_1$ in $\mathcal{T}_1$.

**Proof.** We shall cheat slightly. Di Paola 1966 shows that there is a $\psi_0$ which semirepresents $R_0$ in $\mathcal{T}_0$ and $\omega^n$ in $\mathcal{T}_1$. So let $\psi_1$ uniformly semirepresent $R_1$ in $\mathcal{T}_0$, $\mathcal{T}_1$ and define $\varphi = \psi_0 \land \psi_1$. Q.E.D.

Di Paola’s full result required there to be a recursive interpolant between $R_0$ and $R_1$.

**References**

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M. Hájková and P. Hájek

R. A. di Paola

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