UNIQUE BALAYAGE IN FOURIER TRANSFORMS
ON COMPACT ABELIAN GROUPS

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Abstract. Let $K$ be a compact subset of the compact abelian group $G$ and let $\Lambda$ be a subset of the dual group $\Gamma$. Unique balayage is said to be possible for $(K, \Lambda)$ if, for every $\mu \in M(G)$, there is a unique $\nu \in M(K)$ whose Fourier transform $\hat{\nu}$ agrees on $\Lambda$ with $\hat{\mu}$.

We prove that in order that there be any $K$ with unique balayage possible for $(K, \Lambda)$, $\Lambda$ must belong to the coset ring of $\Gamma$. The converse of this statement is false. Some examples are given for the case where $G$ is the circle group.

1. General results on unique balayage. Let $G$ be a compact abelian group (written multiplicatively) with dual group $\Gamma$ (written additively). If $K \subset G$ is compact and $\Lambda \subset \Gamma$ then, following Beurling [2], balayage is said to be possible for $(K, \Lambda)$, if, for every measure $\mu \in M(G)$ there is a measure $\nu \in M(K)$ with

$$\hat{\mu}(\lambda) = \hat{\nu}(\lambda) \quad \text{for all } \lambda \in \Lambda.$$ 

Here $\hat{\mu}$ denotes the (inverse) Fourier transform

$$\hat{\mu}(\gamma) = \int_G \gamma(x) \, d\mu(x).$$

We would like to thank Professor Colin Graham for acquainting us with the following problem of Professor S. Hartman. Determine whether there are nontrivial examples of sets $K$ and $\Lambda$ with $K \subset T$, the circle group, such that

(1) balayage is possible for $(K, \Lambda)$ and
(2) if $\mu \in M(K)$ and $\hat{\mu}|\Lambda = 0$ then $\mu = 0$.

In any $G$ and $\Gamma$, if these conditions are satisfied we shall say that unique balayage is possible for $(K, \Lambda)$.

We recall that the coset ring of $\Gamma$ is the smallest algebra of sets containing all cosets of subgroups of $\Gamma$. Our basic result is

Theorem 1. Given $\Lambda \subset \Gamma$, in order that there be some $K \subset G$ with unique balayage possible for $(K, \Lambda)$ it is necessary that $\Lambda$ belong to the coset ring of $\Gamma$.

Remark. We shall see in §2 that, when $\Gamma = Z$, there are some infinite $\Lambda$ in
the coset ring for which unique balayage is possible and others for which it is not.

**Proof of Theorem 1.** We first introduce some Banach spaces and reformulate the definition of unique balayage.

$B(\Lambda)$ is the space of restrictions to $\Lambda$ of Fourier transforms and is normed by

$$\|\phi\|_{B(\Lambda)} = \inf \{ \|\mu\| : \mu \in M(G) \text{ and } \hat{\mu}|\Lambda = \phi \}.$$  

$C_\Lambda(G)$ is the space of continuous functions $f$ on $G$ such that $\hat{f}(-\gamma) = 0$ for all $\gamma \in \Gamma \setminus \Lambda$. Here, by $\hat{f}$ we again mean the inverse transform

$$\hat{f}(\gamma) = \int f(x)\gamma(x) \, dx$$

the integral being taken with respect to normalized Haar measure. $C_\Lambda(G)$ is given the uniform norm. It follows from the existence of a version of Cesàro summability on $G$ (see, for example, [1, p. 56]) that $C_\Lambda(G)$ is the uniform closure of the set of trigonometric polynomials with frequencies in $\Lambda$. The pairing of $\phi \in B(\Lambda)$ and $f \in C_\Lambda(G)$ given by

$$\langle \phi, f \rangle = \int_G f(x) \, d\mu,$$

where $\mu \in M(G)$ is any measure with $\hat{\mu}|\Lambda = \phi$, represents $B(\Lambda)$ as the dual space of $C_\Lambda(G)$. (Use the Hahn-Banach and Riesz theorems; see, for example, [4, p. 116] and note that, since $G$ is compact, $AP_\Lambda(G) = C_\Lambda(G)$.) $C(K)$ denotes the space of complex-valued continuous functions on $K$.

We define a bounded linear operator $S: C_\Lambda(G) \to C(K)$ by

$$Sf = f|K. \quad (1)$$

Regarding $M(K)$ as the dual space of $C(K)$, we see that the adjoint $S^*: M(K) \to B(\Lambda)$ is given by

$$S^*\mu = \hat{\mu}|\Lambda. \quad (2)$$

Indeed, setting $\phi = \hat{\mu}|\Lambda \in B(\Lambda)$, we have, for all $f \in C_\Lambda(G)$,

$$\langle \phi, f \rangle = \int_K f(x) \, d\mu(x) = \langle \mu, Sf \rangle$$

so that we must have $\phi = S^*\mu$.

From (2) we see that unique balayage is possible for $(K, \Lambda)$ if and only if $S^*$ is bijective. Supposing this to be the case, it follows [3, p. 479] that $S$ is bijective and, therefore, invertible. Indeed, for $f \in C(K)$, $S^{-1}f$ is the unique $g \in C_\Lambda(G)$ with $g|K = f$.

Define $P: C(G) \to C_\Lambda(G)$ by

$$Pf = S^{-1}(f|K).$$

From the above, it follows that $Pf$ is the unique $g \in C_\Lambda(G)$ such that $g|K = f|K$. In particular, $P$ is a bounded projection of $C(G)$ onto $C_\Lambda(G)$. Theorem 1, therefore, follows from
Theorem 2. A necessary and sufficient condition for the existence of a bounded projection $P$ from $C(G)$ onto $C_\Lambda(G)$ is that $\Lambda$ belong to the coset ring of $\Gamma$.

Proof. This result and the argument establishing it are very similar to a result of Rosenthal [5, Theorem 3] about projections onto translation-invariant subspaces of $L^1(G)$. Some of the details differ, however.

Suppose, first, that such a $P$ exists. Write $f_t$ for the translate, $f_t(x) = f(xt)$, and define an operator $Q$ on $C(G)$ by

$$(Qf)(x) = \int_G (Pf_t)(xt^{-1}) \, dt$$

(integral with respect to Haar measure).

Since $G$ is compact, $t \to f_t$ is norm continuous so that the integrand above is continuous in $t$. If $\gamma \in \Gamma$ then $\gamma \in C(G)$ and $\gamma_t = \gamma(t)\gamma$. Thus

$$(Q\gamma)(x) = \int_G \gamma(t)(P\gamma)(xt^{-1}) \, dt = \gamma(x)(P\gamma)^{-}(-\gamma).$$

Now, $P\gamma \in C_\Lambda(G)$ so that, if $\gamma \not\in \Lambda$ then $Q\gamma = 0$. On the other hand, if $\gamma \in \Lambda$ then $P\gamma = \gamma$ so that $Q\gamma = \gamma$ also. Thus, $Q$ maps trigonometric polynomials into trigonometric polynomials belonging to $C_\Lambda(G)$. Since we clearly have $\|Qf\|_\infty < \|P\| \|f\|_\infty$, the fact that every function in $C(G)$ can be uniformly approximated by trigonometric polynomials now implies that, for $f \in C(G)$, $Qf$ is continuous and, hence, that $Qf \in C_\Lambda(G)$. (Thus, $Q$ is the "obvious" projection of $C(G)$ onto $C_\Lambda(G)$.)

Now, if we regard $Q$ as an operator from $C(G)$ to itself, then $Q^*: M(G) \to M(G)$. If $\mu \in M(G)$, then,

$$(Q^*\mu)^*(\gamma) = \langle \gamma, Q^*\mu \rangle = \langle Q\gamma, \mu \rangle$$

which is $\mu(\gamma)$ or 0 according as $\gamma \in \Lambda$ or $\gamma \not\in \Lambda$.

In particular, if $\delta$ is the unit point mass at 1, and if $\alpha = Q^*\delta$ then $\hat{\alpha}$ is the characteristic function of $\Lambda$ so that $\alpha$ is idempotent. By Cohen's general result on idempotent measures, [6, Theorem 3.1.3], $\Lambda$ must belong to the coset ring of $\Gamma$.

Conversely, if $\Lambda$ belongs to the coset ring of $\Gamma$, then so does $-\Lambda$ and there is a measure $\alpha \in M(G)$ such that $\hat{\alpha}$ is the characteristic function of $-\Lambda$. In this case, if we set $ Pf = f \ast \alpha $, then $P$ is a bounded projection of $C(G)$ onto $C_\Lambda(G)$.

2. Examples in the circle group. In general, if $\Lambda$ is finite, then unique balayage is possible for $(K, \Lambda)$ if and only if $K$ has the same cardinality as $\Lambda$ and the characters in $\Lambda$, considered as functions on $K$, are linearly independent (for then, $C_\Lambda(G)\{K$ has the same dimension as $C(K)$).

For less trivial results, we turn to the case $G = T$, the circle group (realized as $\{z: |z| = 1\}$) so that $\Gamma = Z$. We note that $\Lambda$ belongs to the coset ring of $Z$.
if and only if \( \Lambda \) differs from a periodic set by at most a finite number of elements. We present, first, a class of examples showing that the converse to Theorem 1 fails.

Let \( n > 1 \) be an integer. Let \( B \) be a nontrivial, proper subset of \( \{0, 1, 2, \ldots, n-1\} \), containing \( b \) elements. Let \( n\mathbf{Z} \) be the subgroup of multiples of \( n \). Set

\[
\Lambda = \bigcup_{j \in B} (j + n\mathbf{Z}).
\]

**Theorem 3.** With \( \Lambda \) as above, there is no compact \( K \subset T \) with unique balayage possible for \((K, \Lambda)\).

**Proof.** Let \( \omega = \exp(2\pi i/n) \) and let \( \delta(z) \) denote the unit point mass at \( z \in T \).

Set, for \( j = 0, 1, 2, \ldots, n-1 \),

\[
\mu_j = \frac{1}{n} \sum_{k=0}^{n-1} \omega^{jk} \delta(\omega^k).
\]

Then

\[
\hat{\mu}_j(m) = \frac{1}{n} \sum_{k=0}^{n-1} (\omega^{j+m})^k
\]

which is 1 or 0 according as \( j + m \in n\mathbf{Z} \) or not. Thus, \( \hat{\mu}_j \) is the characteristic function of the coset \(-j + n\mathbf{Z} \). Let \( D \) be the complement of \( B \) in \( \{0, 1, 2, \ldots, n-1\} \) so that \( D \) has \( d = n - b \) elements. In order for \( f \) to belong to \( C_\Lambda(T) \) it is, therefore, necessary and sufficient that

\[
f * \mu_j = 0 \quad \text{for all } j \in D.
\]

More explicitly, \( f \in C_\Lambda(T) \) if and only if, for each \( z \in T \),

\[
\sum_{k=0}^{n-1} \omega^{jk} f(z \omega^{-k}) = 0 \quad \text{for all } j \in D. \tag{3}
\]

Let \( H \) be the subgroup \( \{1, \omega, \omega^2, \ldots, \omega^{n-1}\} \) of \( T \). For each \( z \in T \), (3) is a set of relations which \( f|zH \) must satisfy. Since the \( n \times n \) matrix whose \( jk \)th element is \( \omega^{jk} \) has orthogonal rows the \( d \) relations in (3) are linearly independent. Thus, \( C_\Lambda(T)|zH \) has dimension \( n - d = b \). Since \( C(K)|zH \) has as its dimension the cardinality of \( K \cap zH \), in order that the operator \( S \) (in equation (1)) be bijective it is, therefore, necessary that for each \( z \in T \), \( K \cap zH \) have exactly \( b \) elements. Equivalently, each \( z \in T \) must belong to exactly \( b \) of the translates \( \omega^kK, k = 0, 1, 2, \ldots, n-1 \).

No compact set \( K \) can have this property. Indeed, for each \( W \subset H \) which has \( b \) elements, define

\[
K_W = \bigcap_{z \in W} zK.
\]

If \( W_1 \neq W_2 \) then any element in \( K_{W_1} \cap K_{W_2} \) would belong to at least \( b + 1 \) of the \( \omega^kK \). Thus, the various \( K_W \) are disjoint. Clearly, no \( K_W \) is all of \( T \) but
we must have

\[ T = \bigcup_w K_w. \]

Since each \( K_w \) is compact, this contradicts the connectedness of \( T \) and so proves Theorem 3.

For some infinite sets in the coset ring of \( Z \), unique balayage is possible, as the following example shows. Let \( \Lambda = 2Z \cup \{1\} \) and let \( K = \{e^{i\theta}: 0 < \theta < \pi\} \). Note that a continuous \( g \) belongs to \( C_{2Z}(T) \) if and only if \( g(e^{i\theta}) \) has period \( \pi \).

Suppose \( f \in C(K) \). Set

\[ g(z) = f(z) + \frac{1}{2}(f(-1) - f(1))z \]

for \( z \in K \). Then \( g(1) = g(-1) \) so that \( g \) may be extended to a function \( h \in C_{2Z}(T) \). Then, if

\[ f_1(z) = h(z) - \frac{1}{2}(f(-1) - f(1))z \]

we see that \( f_1 \in C_A(T) \) and that \( f_1|K = f \) so that \( S \) as defined in equation (1) is onto.

On the other hand, suppose \( f \in C_A(T) \) and \( f|A = 0 \). If

\[ h(z) = f(z) - \hat{f}(-1)z \]

then \( h \in C_{2Z}(T) \) so that \( h(e^{i\theta}) \) has period \( \pi \). Thus

\[ \hat{h}(2m) = \int_0^\pi h(e^{i\theta})e^{2im\theta} \frac{d\theta}{\pi} \]

\[ = \int_0^\pi f(e^{i\theta})e^{2im\theta} \frac{d\theta}{\pi} - \int_0^\pi \hat{f}(-1)e^{i(2m+1)\theta} \frac{d\theta}{\pi} = \frac{-2i}{\pi(2m+1)} \hat{f}(-1) \]

because \( f|K = 0 \). If \( \hat{f}(-1) \neq 0 \), these are the Fourier coefficients of a discontinuous step function. Thus \( \hat{f}(-1) \) and, hence, \( \hat{h} \) and \( \hat{f} \), must vanish. Thus, \( f = 0 \) so that unique balayage is possible for \((K, \Lambda)\).

**BIBLIOGRAPHY**


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