MULTIPLIERS ON COMPACT GROUPS

IAN INGLIS

Abstract. We give some sufficient conditions for a function on a compact totally disconnected abelian group to be an $L^p$ Fourier multiplier.

1. Introduction. Let $X$ denote a compact abelian group with a strictly decreasing sequence of open compact subgroups $\{X_n\}_{n=0}^\infty$ such that $\bigcup X_n = X$ and $2 < |X_n| - |X_{n+1}| < b$, where $|S|$ denotes the Haar measure of a set $S$. Let $G$ denote the dual of $X$ and $G_n$ the annihilator of $X_n$ in $G$; thus $\{G_n\}$ is an increasing sequence of open compact subgroups of $G$, $\bigcup G_n = G$, $\bigcap G_n = \{0\}$ and $2 < |G_{n+1}| - |G_n| < b$. We denote by $dx$ and $dg$ the Haar measures on $X$ and $G$ respectively, and assume that these are adjusted so that the inversion theorem holds.

If $\phi \in L^\infty(X)$ then $\phi$ defines a bounded linear operator $T_\phi$ on $L^2(G)$ via the formula

$$T_\phi f \star = \hat{\phi} \hat{f}.$$ 

We say that $\phi$ is an $L^2$ Fourier multiplier. Similarly for $1 < p < \infty$, we say that $\phi$ is an $L^p$ Fourier multiplier, (and write $\phi \in M_p(X)$), if there exists a number $B$ such that

$$\|T_\phi f\|_p \leq B\|f\|_p \quad (1)$$

for all $f$ in $L^p \cap L^2(G)$; we write $\|\phi\|_{M_p}$ for the smallest value of $B$ for which (1) holds. It is well known that $M_p = M_\frac{1}{p}$ (where, as always $(1/p) + (1/p^*) = 1$) and

$$A(X) = M_1(X) \subseteq M_p(X) \subseteq M_q(X) \subseteq M_2(X) = L^\infty(X)$$

when $1 < p < q < 2$ and where $A(X)$ is the space of Fourier transforms of integrable functions with the inherited norm. A wealth of information about multipliers is contained in the book [1] of Edwards and Gaudry.

2. Suppose $1 < \theta < \infty$; for $n > 0$ we define the subgroup $X_n^\theta$ of $X$ by the formula

$$X_n^\theta = X_j \quad \text{where} \quad |X_j| > |X_n^\theta| > |X_{n+1}|.$$ 

Our main result is the following:
Theorem. Suppose \( \phi \) is a function on \( X \) constant on cosets of \( X_n^\theta \) outside \( X_n \). If

\[
|\phi(\chi)| \leq B|X_n|^{{(\theta - 1)/2}} \quad \text{when} \quad \chi \in X_{n-1} \setminus X_n
\]

for some constant \( B \) independent of \( n \), then \( \phi \in M_p(X) \) for \( 1 < p < \infty \).

The proof employed is singular-integral in spirit, although no use is made of Calderón-Zygmund type covering lemmas. We need the following:

Lemma. For \( 1 < \theta < \infty \) we define the subgroup \( G_n^\theta \) of \( G \) by the formula

\[
G_n^\theta = G_j \quad \text{where} \quad |G_j| < |G_n^\theta| < |G_j + 1|.
\]

In other words, \( G_n^\theta \) is the annihilator of \( X_n^\theta \). Suppose \( k \) is an integrable function on \( G \), constant on cosets of \( G_n^\theta \) outside \( G_n^\theta \). If

\[
|k(\chi)| \leq B|X_n|^{{(\theta - 1)/2 + \beta}} \quad \text{when} \quad \chi \in X_{n-1} \setminus X_n
\]

for some \( \beta > 0 \), then

\[
\|k * f\|_\infty \leq B \cdot C \cdot \|f\|_\infty
\]

where \( C \) is a constant independent of \( \|k\|_1 \).

Proof. Fix \( f \) in \( L^\infty \). By translation invariance it suffices to show that, for every \( N > 0 \),

\[
|G_N|^{-1} \int_{G_N} k * f \, d\chi \leq B \cdot C \|f\|_\infty.
\]

Fix \( N \) and write \( f = f_1 + f_2 \) where \( f_1 = f \cdot \xi_{G_n^\theta + 1} \) (\( \xi_S \) denotes the indicator function of the set \( S \)). Then

\[
|G_N|^{-1} \int_{G_N} k * f_1 \, d\chi = |D_N * k * f_1(0)| \quad \text{where} \quad D_n = \xi_{G_n} \cdot |G_n|^{-1}
\]

\[
= \left| \int_{X_n} \hat{k} \hat{f}_1 \, d\chi \right|
\]

\[
\leq B|X_n|^{-1/2 + \beta} \int_{X_n} |\hat{f}_1| \, d\chi \quad \text{by (5)}
\]

\[
\leq B|X_n|^{-1/2 + \beta + 1/2} \left( \int_{X_n} |\hat{f}_1|^2 \, d\chi \right)^{1/2}
\]

by Hölder’s inequality,

\[
\leq B|X_n|^{-1/2 + \beta} \|f_1\|_2
\]

\[
\leq B|X_n|^{-1/2 + \beta} |G_n + 1|^{-1/2} \|f\|_\infty
\]

by the definition of \( f_1 \),

\[
\leq B \cdot b^{\theta/2} |X_n| \|f\|_\infty.
\]

Now clearly,
\[ |G_N|^{-1} \int_{G_N} k \ast f_2 \, dx \]

\[ \leq |G_N|^{-1} \int_{G_N} |k \ast f_2 - |G_{N+1}|^{-1} \int_{G_{N+1}} k \ast f_2| \, dx \]

\[ + |G_{N+1}|^{-1} \int_{G_{N+1}} k \ast f_2 \]

But

\[ |G_N|^{-1} \int_{G_N} k \ast f_2 - |G_{N+1}|^{-1} \int_{G_{N+1}} k \ast f_2 \, dx \]

\[ \leq |G_N|^{-1} \int_{G_N} |k \ast f_2 - |G_{N+1}|^{-1} \int_{G_{N+1}} |k \ast f_2 - \sigma| \, dx \]

\[ \text{where } \sigma = \int_G k(-y) f_2(y) \, dy. \quad (7) \]

The second term on the right of (7) is equal to

\[ |G_{N+1}|^{-1} \int_{G_{N+1}} dx \left( \left| \int_{G \setminus G_{N+1}} (k(x-y) - k(-y)) f_2(y) \, dy \right| \right) = 0 \]

since \( x \in G_{N+1} \) and \( k \) is constant on the cosets of \( G_{N+1} \) outside \( G_{N+1}^\theta \). The same argument shows that the first term on the right of (7) is also zero, so

\[ |G_N|^{-1} \int_{G_N} k \ast f_1 \, dx \]

Now write \( f_2 = f_3 + f_4 \) where \( f_3 = f_2 \cdot \xi_{G_{N+1}} \). The argument used to estimate \( |G_N|^{-1} \int_{G_N} k \ast f_1 \, dx \) shows that

\[ |G_{N+1}|^{-1} \int_{G_{N+1}} k \ast f_3 \, dx \leq B \cdot b^{\theta/2} |X_{N+1}|^\beta \| f \|_\infty, \]

and the argument used to estimate \( |G_N|^{-1} \int_{G_N} k \ast f_2 \, dx \) shows that

\[ |G_{N+1}|^{-1} \int_{G_{N+1}} k \ast f_4 \, dx \leq |G_{N+2}|^{-1} \int_{G_{N+2}} k \ast f_4 \, dx \]

We may suppose without loss of generality that \( \hat{k} = 0 \) on \( X_M \) for some large \( M \); thus a continuation of the above argument leads to the estimate

\[ |G_N|^{-1} \int_{G_N} k \ast f \, dx \leq b^{\theta/2} \cdot B \cdot \| f \|_\infty \left( \sum_{n=0}^{\infty} |X_n|^\beta \right) \]

\[ \leq b^{\theta/2} \cdot B \cdot \| f \|_\infty \sum_{n=0}^{\infty} 2^{-n\beta} \]

\[ \leq b^{\theta/2} \cdot B \cdot (1 - 2^{-\beta})^{-1} \| f \|_\infty, \]

which proves the lemma.
Proof of Theorem. Set $\Theta(x) = |X_n^{(\theta - 1)/2}$ when $x \in X_{n-1} \setminus X_n$, and consider the family of operators $U_z$ on $L^2(G)$ defined by

$$(U_z f) = \hat{f} \cdot \Theta^{-z + \beta}$$

where $0 < \text{Re} z < 1$ and $\beta > 0$. It is easy to check that, by virtue of (3), the mapping $z \to U_z$ is uniformly bounded and strongly continuous in the strip $0 < \text{Re} z < 1$ and analytic in $0 < \text{Re} z < 1$, to the space of bounded linear operators on $L^2(G)$.

It follows immediately from (3) that

$$\|U_{1+\beta}f\|_2 \leq B \cdot \|f\|_2.$$  \hspace{1cm} (8)

Furthermore it follows from the above lemma that

$$\|U_{\Theta}f\|_\infty \leq B \cdot C \cdot \|f\|_\infty.$$  \hspace{1cm} (9)

(This amounts essentially to the observation that a function constant on cosets of $X_n^\theta$ outside $X_n$ has Fourier transform constant on cosets of $G_n$ outside $G_n^\theta$.) An application of Stein's interpolation theorem, [6, p. 205], shows that

$$\phi \cdot \Theta^{-i + \beta} \in M_{2/t}(X)$$

where $0 < t < 1$, and $\beta > 0$. To see that $\phi \in M_p$ for all $p$ in $(1, \infty)$ fix $p$ in $(2, \infty)$ and set $t = 2/p$. Then

$$\phi \Theta^{-2/p + \beta} \in M_p(X),$$

in particular when $\beta = 2/p$. The proof is complete.

Virtually the same proof yields:

Corollary. Under the same conditions as stated in the theorem

$$\phi \cdot \Theta^{-t} \in M_p(X) \quad \text{when} \quad 2/(2 - t) < p < 2/t.$$  \hspace{1cm} (10)

It is not difficult, using the ideas in [2], to construct examples of functions $\phi$ such that $\phi \cdot \Theta^{-t} \in M_p$ when $p > 2/t$. The interesting case is, of course, when $p = 2/t$—see remark (b) below.

3. Remarks. (a) The condition that $|X_n| \cdot |X_{n+1}|^{-1} < b$ is not really necessary. If $b_n = |X_n| \cdot |X_{n+1}|^{-1}$ is such that $b_n \uparrow \infty$ and $\Sigma b_n^2 < \infty$, then the above proof is easily adapted to show that:

If $\phi$ is a function constant on cosets of $X_n$ outside $X_n$ then $|\phi(x)| \leq B \cdot b_n^{-(1/2 + \beta)}$ when $x \in X_n \setminus X_{n+1}$ for some $\beta > 0$ implies that $\phi \in A(X)$, and $|\phi(x)| \leq B \cdot b_n^{-1/2}$ when $x \in X_n \setminus X_{n+1}$ implies that $\phi \in M_p(X)$ for $1 < p < \infty$.

This complements some results of Spector [5]. Further results may be obtained by considering subsequences $\{X_{n_k}\}$ of $\{X_n\}$.

(b) There is a natural definition of $\text{BMO}(G)$, the space of functions of bounded mean oscillation on $G$, see [2]. It would be interesting to know if the functions $\phi$ satisfying the hypotheses of the theorem are $L^\infty \to \text{BMO}$ "multipliers". A positive answer would imply that $\phi \cdot \Theta^{-t} \in M_{2/t}$ when $0 < t < 1$. 

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(c) Let $Z$ denote the group of integers, and $Z_n$ the subgroup $2^n \cdot 2^{n-1} \cdot \ldots \cdot 2^1 Z$, where $n \geq 0$. Suppose $\phi$ is a function constant on cosets of $Z_{n+1}$ in $Z_n \setminus Z_{n+1}$. If $|\phi(m)| < B \cdot 2^{-(n+1)(\beta + 1/2)}$ when $m \in Z_n \setminus Z_{n+1}$ for some $\beta > 0$ then $\phi$ is a Fourier-Stieltjes transform, and if $|\phi(m)| < B \cdot 2^{-(n+1)/2}$ when $m \in Z_n \setminus Z_{n+1}$ then $\phi \in M_p(Z)$ for $1 < p < \infty$. To see this argue as follows: Let $X = A_\alpha$, the $\alpha$-adic integers, where $\alpha = (2, 4, 8, 16, \ldots)$ (see [3, §10]), and let $X_n = \{x = (x_j)_{j=0}^{\infty} \in X : x_j = 0$ when $0 < j < n - 1\}$. It is easily seen that $|X_n| = (2^n \cdot 2^{n-1} \cdot \ldots \cdot 2^1)^{-1}$. $X$ has an (algebraic!) subgroup isomorphic to $Z$, namely the group generated by the element $(1, 0, 0, 0, \ldots)$, which we also denote by $Z$. Some rather tedious calculations show that the cosets of $Z_{n+1}$ in $Z_n \setminus Z_{n+1}$ sit inside the cosets of $X_n$ in $X_n \setminus X_{n+1}$; hence $\phi$ is the restriction of a function $\Phi$ on $X$, satisfying the hypotheses in (a) above. The result now follows immediately from well-known results about restrictions of Fourier multipliers to subgroups, see for example Saeki [4, Corollary 4.6].

REFERENCES


ISTITUTO MATEMATICO, UNIVERSITÀ DI MILANO, 20133 MILANO, ITALY