MARKOV PROPERTY OF EXTREMAL LOCAL FIELDS

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\textbf{Abstract.} We show that extremal local field on \((E, \mathcal{E})^T\), with \(T = \mathbb{Z}\) or \(\mathbb{R}\) and \((E, \mathcal{E})\) standard, possesses the Markov property. This result generalizes that of F. Spitzer in the case \(T = \mathbb{Z}\), \(E\) countable and a result of G. Royer and M. Yor on extremal measures associated to certain diffusion processes.

\section{I. Introduction.} Let \(T = \mathbb{R}\) or \(\mathbb{Z}\), let \((E, \mathcal{E})\) be a standard Borel space, \((\Omega, \mathcal{F}, P)\) a probability space and \((X_t)^t \in T\) a stochastic process taking values in \(E\). If \(\Lambda\) is a subset of \(T\), we note \(\mathcal{F}_\Lambda\) the sub-\(\sigma\)-algebra of \(\mathcal{F}\) generated by \(X_t, t \in \Lambda\).

The process \((X_t)\) is called a \textit{local Markov process} (or \textit{local Markov field}) if for every \(a < b\) in \(T\), every positive \(\mathcal{E}_a \upharpoonright (a, b]\)-measurable function \(f\) we have
\[
E\left[ f \mathbb{1}_{(-\infty, a \cup (b, +\infty]} \right] = E\left[ f \mathbb{1}_{(a, b]} \right] \quad P\text{-a.s.}
\]

The process \((X_t)\) is called a \textit{Markov process} (or Markov field) if for every \(t\) in \(T\), every positive \(\mathcal{E}_{[t, +\infty]}\) measurable function \(f\) we have
\[
E\left[ f \mathbb{1}_{(-\infty, t]} \right] = E\left[ f \mathbb{1}_{t} \right] \quad P\text{-a.s.}
\]

The property (0) is called \textit{local Markov property}, the property (1) is called \textit{Markov property}, it is known that Markov property implies local Markov property (see for example [1]) but the converse is false (cf. [4], [5]).

The purpose of this paper is to show that under quite general conditions the local Markov property and the hypothesis \(\mathcal{F}_\infty = \{\emptyset, \Omega\}\) \(P\)-a.s. imply the Markov property, where \(\mathcal{F}_\infty = \bigcap \mathcal{F}_{[t, +\infty]}\) is the asymptotic \(\sigma\)-algebra of the process (we note similarly \(\mathcal{F}_{-\infty} = \bigcap \mathcal{F}_{[t, -\infty]}\)).

This result has been proved under the hypotheses \(E\) countable and \(T = \mathbb{Z}\) by F. Spitzer (cf. [5]). In the case \(E = \mathbb{R}, T = \mathbb{R}\), G. Royer and M. Yor have proved this result for certain diffusion processes associated to one-dimensional quantum fields (cf. [4]).

\section{II. Sufficient conditions for Markov property.}

\textbf{Notations.} We take the path space \((\Omega, \mathcal{F}) = (E, \mathcal{E})^T, X_t(\omega) = \omega, \forall \omega \in E^T, \forall t \in T\). If \(\mu\) is a probability on \((\Omega, \mathcal{F}), \Lambda \subset T\), we note \(\mu_{\Lambda}\) the
restriction of $\mu$ on $\mathcal{G}_A$ and $E_\mu[ g|\mathcal{G}_A]$ the conditional expectation of a random variable $g$ on $\Omega$ with respect to the sub-$\sigma$-algebra $\mathcal{G}_A$.

If a probability $\nu$ on a $\sigma$-algebra $\mathcal{B}$ is absolutely continuous with respect to another probability $\nu'$ on $\mathcal{B}$, we write $\nu \ll \nu'$.

We say that $\mu$ is a local Markov field (resp. Markov field) if $(X_t)$ is a local Markov process (resp. Markov process) under $\mu$.

The main result of the paper is

**Theorem 1.** Let $T = \mathbb{R}$ or $\mathbb{Z}$, $(E, \mathcal{S})$ a standard Borel space. Let $\mu$ be a local Markov field on $(\Omega, \mathcal{G})$ such that

(i) For every $t \in T$, there exist $t' < t < t''$ in $T$ and $\sigma$-finite measures $\nu_t, \nu_t', \nu_t''$ on $(E, \mathcal{S})$ such that

$$\mu_{t', t''} \ll \nu_t \otimes \nu_t' \otimes \nu_t''.$$

(ii) $\mathcal{G}_\infty$ (resp. $\mathcal{G}_{-\infty}$) is trivial for $\mu$.

Then $\mu$ is a Markov process.

We give the proof for the case $T = \mathbb{R}$ (the case $T = \mathbb{Z}$ is similar) under the hypothesis $\mathcal{G}_\infty$ trivial (for the case $\mathcal{G}_{-\infty}$ trivial, we use the additional remark that the Markov property is symmetric with respect to the direction of time).

**Proof.** Let $t \in T$, $t' < t < t''$, $\nu_t, \nu_t', \nu_t''$ as in (i), let $k$ be a positive integer greater than $t''$, $S = \{t', t, t'', k, k + 1, k + 2, \ldots \}$

(a) On $(E, \mathcal{S})^S = (E, \mathcal{S})^{t'} \times (E, \mathcal{S})^{t} \times (E, \mathcal{S})^{t''}$ we have $\mu_S \ll \nu_t \otimes \mu_{S^{t'}}^{t'}. \mu_{S^{t''}}^{t''}$.

Let $(E, \mathcal{S})^S = (E, \mathcal{S})^{(t')} \times (E, \mathcal{S})^{(t)} \times (E, \mathcal{S})^{(t'')} \times (E, \mathcal{S})^{(k, k+1, \ldots)}$, $(E, \mathcal{S})^S = (X, \mathcal{X}) \times (Y, \mathcal{Y}) \times (Z, \mathcal{Z})$.

Consider the measure $\mu_{(t', t'')}^{(t', t'')}$ on $X \times Y = E^{(t')} \times E^{(t'')}$ and let $\mu^{(x, x')}(F)$, for $(x_t, x_t') \in Y, F \in \mathcal{Y}$ be the transition probability on $Y \times \mathcal{Y}$ such that

$$\mu^{(x, x')} = \int_Y \mu^{(x, x')} d\mu_{(t', t'')}^{(t', t'')}(x_t, x_t').$$

(cf. [3, Proposition V.4.4, p. 183]).

And similarly on $X \times (Y \times Z)$ let $\mu^{(x_t, x_t', x_k, x_{k+1}, \ldots)}(F)$, for $(x_t, x_t', x_k, x_{k+1}, \ldots) \in Y \times Z, F \in \mathcal{Y}$ be the transition probability on $(Y \times Z) \times \mathcal{Y}$ such that

$$\mu_S = \int_{Y \times Z} \mu^{(x_t, x_{t'}, x_k, x_{k+1}, \ldots)} d\mu_{S^{t'}}^{(t', t'')}(x_t, x_{t'}, x_k, \ldots).$$

By local Markov property (0), we have

$$\mu^{(x_t, x_{t'}, x_k, \ldots)} = \mu^{(x_t, x_{t'})},$$

for $\mu_{S^{t'}}^{(t', t'')}$-almost all $(x_t, x_{t'})$.

But by (i), we have $\mu_{t', t''} \ll \nu_t \otimes \nu_t' \otimes \nu_t''$ and $\mu_{t', t''} \ll \nu_t \otimes \nu_t''$, so relation (2) implies

$$\mu^{(x_t, x_{t'})} \ll \nu_t,$$

for $\mu_{S^{t'}}^{(t', t'')}$-almost all $(x_t, x_{t'})$.

Comparing (4) and (5), we deduce that

$$\mu^{(x_t, x_{t'}, x_k, \ldots)} \ll \nu_t,$$

for $\mu_{S^{t'}}^{(t', t'')}$-almost all $(x_t, x_{t'}, x_k, \ldots)$. 

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Combining this relation and (3), we obtain

$$
\mu_S \ll \nu \times \mu_{S \setminus \tau}.
$$

(b) We have \( \mathcal{G}_t = \lim_{n \to \infty} \mathcal{G}_t \cap (\bigvee_{m \geq n} \mathcal{G}_m) \): Let \((X, \mathcal{X}) = (E, \mathcal{G})^{(t)}, (W, \mathcal{W}) = (E, \mathcal{G})^{(\tau)}\), then \((E, \mathcal{G})^{(\tau)} = (X, \mathcal{X}) \times (W, \mathcal{W})\).

Let \( \mathcal{B}_n = \bigvee_{m \geq n} \mathcal{G}_m \), for \( n > k \), we shall show that \( \lim_{n \to \infty} \mathcal{G}_t \cap \mathcal{B}_n = \mathcal{G}_t \), \( \mu_S \)-a.e. But we remark that \( -\mathcal{G}_t = \mathcal{X}, -\mathcal{B}_n \subseteq \mathcal{W}, \forall n > k \) and \( \lim_{n \to \infty} \mathcal{B}_n = \{\emptyset, \Omega\}, \mu_{S \setminus \tau} \)-a.e. \( \mathcal{X} \) and \( \mathcal{W} \) are independent with respect to \( \nu, \mathcal{G} \times \mu_{S \setminus \tau} \).

Therefore

$$
\lim_{n} \mathcal{G}_t \cap \mathcal{B}_n = \mathcal{G}_t, \quad \nu \times \mu_{S \setminus \tau} \text{-a.e.}
$$

and also \( \mu_S \)-a.e. since \( \mu_S \ll \nu \times \mu_{S \setminus \tau} \).

(c) Now let \( f > 0, \mathcal{G}_{\alpha, \beta} \) measurable with \( t < \alpha < \beta \), then (1) is satisfied:

Let \( n > \beta \) with \( n \) positive integer, by local Markov property (0), we have

$$
E_{\mu} [ f | \mathcal{G}_{-\infty, n} \cup \mathcal{G}_{n+1, +\infty} ] = E_{\mu} [ f | \mathcal{G}_{(t,n)} ]
$$

$$
= E_{\mu} [ f | \mathcal{G}_t \cap \mathcal{G}_{(t,n+1, \ldots}) ].
$$

So by (b) and martingale theorem, we have

$$
\lim_{n \to \infty} E_{\mu} [ f | \mathcal{G}_{-\infty, n} \cup \mathcal{G}_{n+1, +\infty} ] \to E_{\mu} [ f | \mathcal{G}_t ].
$$

Taking the conditional expectation with respect to \( \mathcal{G}_{-\infty, t} \) of both sides, we obtain

$$
E_{\mu} [ f | \mathcal{G}_{-\infty, t} ] = E [ f | \mathcal{G}_t ], \quad \mu \text{-a.e.}
$$

(d) From (c) we deduce (1) since the \( \mathcal{G}_{\alpha, \beta} \), \( t < \alpha < \beta \), generate \( \mathcal{G}_{t, +\infty} \).

**Remark.** It is well known that the conditions \( \mathcal{G}_t \setminus \{\emptyset, \Omega\} \) do not imply \( \mathcal{G}_t \cup \mathcal{B}_n \setminus \mathcal{G}_t \) without extra hypotheses.

### III. Application to statistical mechanics and one dimensional field theory.

**Local Markov specifications.** Let \( T = \mathbb{R} \) or \( \mathbb{Z} \), \((E, \mathcal{G})\) be a standard Borel space, \((\Omega, \mathcal{F}) = (E, \mathcal{G})^T\).

We call local Markov specifications (see [2]) the given of a family \( \pi = (\pi_{a,b})_{a < b} \) of transition probabilities \( \pi_{a,b}(\omega, A) (\omega \in \Omega, A \in \mathcal{G}) \) such that

(i) For every \( t \in T \) there exist \( a < t < b \), a \( \sigma \)-finite measure \( \nu \) on \((E, \mathcal{G})\) such that the restriction of \( \pi_{a,b} \) to \( \mathcal{G}_t \) is absolutely continuous with respect to \( \nu \), \( \forall \omega \in \Omega \).

(ii) \( \pi_{a,b} \) is \( \mathcal{G}_{-\infty, a} \cup [b, +\infty[ \)-measurable, \( \forall A \in \mathcal{G} \).

(iii) \( \pi_{a,a} \) is \( \mathcal{G}_{a, a} \)-measurable, for \( A \in \mathcal{G}_{a,a} \).

(iv) If \( [a, b] \subset [c, d] \), we have \( \pi_{[c,d]}(\omega, A) = \int_{\Omega} \pi_{[c,d]}(\omega, d\omega') \pi_{[a,b]}(\omega', A) \), \( \forall A \in \mathcal{G}, \forall \omega \in \Omega \).

We call local Markov field (or Gibbs state) specified by \( \pi \) every probability \( \mu \) on \((\Omega, \mathcal{F})\) satisfying

$$
E_{\mu} [ A | \mathcal{G}_{-\infty, a} \cup [b, +\infty[ ](\omega) = \pi_{[a,b]}(\omega, A), \quad \mu \text{-a.e., } \forall A \in \mathcal{G}_{[a,b]}.
$$
The set $\mathcal{G}(\pi)$ of all local Markov fields specified by $\pi$ is a convex set which is possibly empty.

A local Markov field $\mu \in \mathcal{G}(\pi)$ is extremal iff

$$\bigcap_n \bigvee_{|r| > n} \mathcal{G}_r = \{\emptyset, \Omega\}, \mu\text{-a.e. (cf. [2])}.$$ 

Theorem 1 implies

**Theorem 2.** Let $(\pi_{[a,b]})_{a < b}$ be local Markov specifications. Every extremal point of $\mathcal{G}(\pi)$ has the Markov property.

**Remarks.**

1. When $T = \mathbb{Z}$, $E$ countable, this result implies Theorem 6 of F. Spitzer (cf. [5]).
2. This result also implies Theorem 3.9 of G. Royer and M. Yor (cf. [4]).
3. It should be interesting to know whether similar results hold when $T = \mathbb{Z}^d$ or $\mathbb{R}^d$ with $d \geq 2$.
4. The characterization of Gibbs states of $\mathcal{G}(\pi)$ possessing the Markov property (other than extremal Gibbs states) is given in [1].

**Bibliography**


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