APPROXIMATION OF INDUCED AUTOMORPHISMS
AND SPECIAL AUTOMORPHISMS

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Abstract. A class of measure-preserving invertible point transformations which admit approximations is defined. If $T$ is an automorphism which admits an approximation, conditions are given such that an induced automorphism and a special automorphism over $T$ again admit an approximation.

1. Preliminaries. Let $(X, F, \mu)$ be a measure-space isomorphic to the unit interval with Lebesgue measure. A measure-preserving invertible point transformation of $X$ is called an automorphism of $(X, F, \mu)$.

A finite ordered collection $\xi = \{C_i: 1 < i < q\}$ of pairwise-disjoint measurable sets in $X$ is called a partition. If the union of members of $\xi$ is $X$, then $\xi$ is called a partition of $X$. If $A \in F$, we write $A < \xi$ if $A$ is a union of members of $\xi$. If $\eta = \{B_j: 1 < j < p\}$ is a partition, we write $\eta < \xi$ if $B_j < \xi$ for $1 < j < p$. If $0 < \beta < 1$ we write $\xi \supset \beta \eta$ if each $B_j \in \eta$ contains a $C_i \in \xi$ with $\mu(C_i) > \beta \mu(B_j)$.

Let $\varepsilon_X$ denote the partition of $X$ into points. A sequence of partitions $\{\xi(n)\}$ converges to the unit partition, written $\xi(n) \to \varepsilon_X$, if for each $A \in F$, $\mu(A \Delta A(\xi(n))) \to 0$ as $n \to \infty$, where $A(\xi(n)) < \xi(n)$ and is such that $\mu(A \Delta A(\xi(n)))$ is a minimum.

The following definitions are due to Katok and Stepin [4] and Chacon [1], respectively.

Definition 1. An automorphism $T$ is said to admit a cyclic approximation by periodic transformations of the first kind (cyclic a. p. t. 1) with speed $f(n)$, where $f(n)$ is a sequence of real numbers decreasing to zero, if there exists a sequence of partitions $\{\xi(n)\}$, $\xi(n) = \{C_i(n): 1 < i < q(n)\}$ such that:

1. $\xi(n) \to \varepsilon_X$; and
2. $\Sigma_{i=1}^{q(n)} \mu(TC_i(n) \Delta C_{i+1}(n)) < f(q(n))$ where $C_{q(n)+1}$ means $C_1(n)$.

Definition 2. An automorphism $T$ is said to admit a simple approximation if there exists a sequence of partitions $\{\xi(n)\}$, $\xi(n) = \{C_i(n): 1 < i < q(n)\}$ such that:

1. $\xi(n) \to \varepsilon_X$; and
2. $TC_i(n) = C_{i+1}(n), 1 < i < q(n) - 1$.

It is known that automorphisms which admit a simple approximation are ergodic and have simple spectrum.

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2. Induced automorphisms and special automorphisms. Let $T$ be an automorphism of $(X, F, \mu)$. Let $A \in F$.

**Definition 3.** The induced automorphism $T_A: A \to A$ is defined by

$$T_A x = T^k x, \quad x \in A,$$

where $k$ is the least positive integer such that $T^k x \in A$ (neglecting sets of measure zero).

Denote by $\mathbb{N}$ the set of nonnegative integers, and by $\mathbb{R}^+$ the set of positive real numbers. Let $f: X \to \mathbb{N}$ be an integrable function. Put $B(k, n) = \{(x, n): x \in X, f(x) = k\}$. Put $X(f) = \bigcup_{k \geq 0} \bigcup_{n = 0}^k B(k, n)$ and identify $X$ with the set $\bigcup_{k \geq 0} B(k, 0)$. We may regard each set $B(k, n)$, $0 < n < k$, as a copy of $B(k, 0)$. Consequently, we may extend $\mu$ to $X(f)$ and form a normalised measure $\mu'$ on $X(f)$ in the obvious way.

**Definition 4.** Let the transformation $T_f$ on $X(f)$ be defined by

$$T_f(x, n) = (x, n + 1), \quad 0 < n < f(x),$$

$$T_f(x, f(x)) = (Tx, 0).$$

$T_f$ is called the special automorphism over $T$ built under the function $f$.

If $f$ is the characteristic function of a set $A \in F$, then the special automorphism $T_f$ is denoted $T_A$ and is called a primitive automorphism over $T$.

Order the sets $B(k, n)$, $k > 0$, $0 < n < k$, lexicographically.

**Definition 5.** Let $\xi$ be a partition in $X$ such that every element of $\xi$ is contained in exactly one of the sets $B(k, 0)$ for some $k$. Then $\xi^f$ is the partition in $X(f)$ consisting of the elements $C \in \xi$, together with, for each $C \in \xi$, where $C \subset B(k, 0)$, a copy of $C$ in each of the sets $B(k, n)$, $0 < n < k$. The ordering on $\xi^f$ is that inherited from the sets $B(k, n)$, $k > 0$, $0 < n < k$.

It is easily seen that if $\xi(n) \to \varepsilon_X$, then $\xi^f(n) \to \varepsilon_X(f)$.

Recall that the measure algebra $(F, \mu)$ is a complete metric space with respect to the metric $d$ given by

$$d(A, B) = \mu(A \Delta B), \quad A, B \in F.$$ 

Goodson [3] has proved the following two theorems for induced and primitive automorphisms.

**Theorem 1.** Let $T: X \to X$ be an automorphism which admits a simple approximation; then there is a collection of subsets of $X$, dense in $F$, such that the automorphisms $T^A$ and $T_A$ on any one of these sets also admit a simple approximation.

**Theorem 2.** Let $T: X \to X$ admit a simple approximation with respect to a sequence of partitions $\{\xi(n)\}$, $\xi(n)$ having $q(n)$ elements, and suppose $A \in F$, with $\mu(A) > 0$, can be approximated by sets $A(n) \subset A$ with $A(n) \leq \xi(n)$ and such that $q(n)\mu(A \setminus A(n)) \to 0$ as $n \to \infty$. Then $T^A$ and $T_A$ both admit a simple approximation.
3. Main results. We show below that if $T$ admits a cyclic a.p.t. I with fast enough speed then both $T_A$ and $T^A$ admit a simple approximation, where $A$ is as in Theorem 2. We require the following lemma.

**Lemma 1.** Let $\{\xi(n)\}$ be a sequence of partitions with $\xi(n) \to \varepsilon_X$. Let $\{\xi(n)\}$ be a sequence of partitions and let $\beta(n)$ be an increasing sequence of positive real numbers converging to 1 such that $\xi(n) \supset \beta(n) \xi(n)$, $n > 1$. Then $\xi(n) \to \varepsilon_X$.

**Proof.** Let $A \in F$ with $\mu(A) > 0$. Since $\xi(n) \to \varepsilon_X$, there is a sequence of sets $A(\xi(n)) \subset \xi(n)$ such that $\mu(A \triangle A(\xi(n))) \to 0$ as $n \to \infty$.

Let $D(n) = \bigcup \{C(n) \in \xi(n): C(n) \subset A(\xi(n))\}$; then

$$\mu(A(\xi(n)) \triangle D(n)) \leq (1 - \beta(n)) \mu(\bigcup \{C(n): C(n) \in \xi(n)\}) < 1 - \beta(n).$$

Consequently,

$$\mu(A \triangle D(n)) \leq \mu(A \triangle A(\xi(n))) + \mu(A(\xi(n)) \triangle D(n)) \to 0 \quad \text{as } n \to \infty.$$

Hence,

$$\mu(A \triangle A(\xi(n))) \leq \mu(A \triangle D(n)) < 0 \quad \text{as } n \to \infty.$$

The proof of the following result is based on techniques shown in [2].

**Theorem 3.** Let $T: X \to X$ be an automorphism admitting a cyclic a.p.t. I with speed $f(n) = o(1/n^2)$ with respect to a sequence of partitions $\xi(n) \to \varepsilon_X$, $\xi(n) = \{C_i(n): 1 \leq i \leq q(n)\}$. Let $A \in F$ be approximated by sets $A(n) \subset A(n) \subset A$ such that $q(n) \mu(A \setminus A(n)) \to 0$ as $n \to \infty$. Then $T_A$ and $T^A$ admit a simple approximation.

**Proof.** For each $i, 1 \leq i \leq q(n)$, we can write $C_i(n) = F_i(n) \cup G_i(n)$, where $TG_i(n) \subset C_{i+1}(n)$ and $TF_i(n) \cap C_{i+1}(n) = \emptyset, 1 \leq i \leq q(n)$. $C_{q(n)+1}(n)$ means $C_0(n)$.

Let $C_i'(n) = C_i(n) \cap \bigcap_{i=1}^{q(n)-1} T^{-i-1} G_i(n)$ and put

$$T^{-1} C_i'(n) = C_i'(n), \quad 1 \leq i \leq q(n).$$

As in [2], it can be shown that

$$\mu(C_i'(n) \setminus C_i(n)) \leq \frac{1}{2} f(q(n)).$$

Thus,

$$\mu(C_i'(n)) > \mu(C_i(n)) - \frac{1}{2} f(q(n)) = \mu(C_i(n))(1 - o(1/q(n))).$$

Consequently, by Lemma 1, $\xi(n) \to \varepsilon_X$, where $\xi(n) = \{C_i'(n): 1 \leq i \leq q(n)\}$, and $T$ admits a simple approximation with respect to the sequence $\{\xi'(n)\}$.

Let $A'(n) = \bigcup \{C_i(n) \in \xi(n): C_i(n) \subset A(n)\}$. Then $A'(n) \subset A$ and

$$\mu(A \setminus A'(n)) \leq \mu(A \setminus A(n)) + \mu(A(n) \setminus A'(n))$$

$$< \mu(A \setminus A(n)) + \frac{1}{2} q(n) f(q(n)) \to 0 \quad \text{as } n \to \infty.$$

Furthermore,

$$q(n) \mu(A \setminus A'(n)) \to 0 \quad \text{as } n \to \infty.$$
Hence by Theorem 2, \( T_A \) and \( T^4 \) admit a simple approximation.

Next, we give results for special automorphisms analogous to Theorems 1 and 2. From our results we are able to obtain a slightly weaker version of Theorem 2 for primitive automorphisms as a corollary.

**Theorem 4.** Let \( T: X \to X \) be an automorphism admitting a simple approximation with respect to a sequence of partitions \( \xi(n) \to \varepsilon_X \). Let \( f: X \to \mathbb{N} \) be integrable. Let \( \xi(n) = \{ C_i(n): 1 \leq i < q(n) \} \). Suppose that for each \( C_i(n) \in \xi(n), 1 \leq i < q(n) \), there exists \( k_i(n) \in \mathbb{N} \) such that

\[
\mu \{ x \in C_i(n): f(x) = k_i(n) \} > (1 - \delta(n)) \mu(C_i(n))
\]

where \( \delta(n) \) is a sequence of real numbers with \( \delta(n) = o(1/q(n)) \). Then \( T_f \) admits a simple approximation.

**Proof.** Let \( B_i(n) = f^{-1}(k_i(n)) \). Put

\[
D_i(n) = \bigcap_{i=0}^{q(n)-1} T^{-i}(C_{i+1}(n) \cap B_{i+1}(n)).
\]

Then

\[
\mu(D_i(n)) \geq \mu(C_i(n)) - q(n)\delta(n) \mu(C_i(n)) = \mu(C_i(n))(1 - q(n)\delta(n)). \tag{1}
\]

Put \( D_i(n) = T^{-1}D_i(n), 1 \leq i < q(n), \) and put \( \eta(n) = \{ D_i(n): 1 \leq i < q(n) \} \). Then \( \eta(n) \supset (1 - q(n)\delta(n))\xi(n) \) by (1), and consequently \( \eta(n) \to \varepsilon_X \) by Lemma 1. Hence \( \eta_f(n) \to \varepsilon_X(f) \) and \( T_f \) admits a simple approximation with respect to \( \{ \eta_f(n) \} \).

In the case where \( f \) is the characteristic function of a set \( A \), Theorem 4 has the following weaker form of Goodson's result as a corollary.

**Corollary 1.** Let \( T: X \to X \) be an automorphism which admits a simple approximation with respect to a sequence of partitions \( \xi(n) \to \varepsilon_X \). Let \( A \) be a measurable set approximated by sets \( A(n) \subset A(n), A(n) \prec A, \) in the sense that \( q(n)^2\mu(A \setminus A(n)) \to 0 \) as \( n \to \infty \). Then \( T^4 \) admits a simple approximation.

**Proof.** Let \( f(x) = 1, x \in A, = 0, x \in X \setminus A \). Let \( C(n) \in \xi(n) \). If \( C(n) \subset A(n) \), then clearly \( \mu \{ x \in C(n): f(x) = 1 \} = \mu(C(n)) \). If \( C(n) \subset X \setminus A(n) \), then

\[
\mu \{ x \in C(n): f(x) = 0 \} = \mu(C(n)) - \mu(C(n) \cap A) \\
\geq \mu(C(n)) - \mu(A \setminus A(n)) \\
= \mu(C(n)) - o(1/q(n)^2) \\
= \mu(C(n))(1 - o(1/q(n))).
\]

Hence by Theorem 4, \( T_f = T^4 \) admits a simple approximation.

We now give a density result for special automorphisms which is analogous to that of Theorem 1.
Theorem 5. The collection of functions \( f: X \to \mathbb{N} \) which satisfy the conditions of Theorem 4 is dense in the set of integrable functions \( g: X \to \mathbb{N} \) with the \( L_1 \) topology.

Proof. Suppose that \( g: X \to \mathbb{N} \) is integrable, and takes values \( k_1 < k_2 < \ldots \). Put \( G_i = g^{-1}(k_i), i > 1 \). We assume, without loss of generality, that \( \mu(G_i) > 0, i > 1 \). Now choose \( m \in \mathbb{N} \) sufficiently large so that, given \( \varepsilon > 0 \),

\[
\int_{X \setminus \bigcup_{i=1}^m G_i} g \, d\mu < \frac{\varepsilon}{4}.
\]

(2)

Since \( \xi(n) \to \varepsilon_X \), we can find \( n \) such that there are \( \xi(n) \)-sets \( E_i(n) < \xi(n) \), \( 1 < i < m \), which are pairwise disjoint and \( \mu(E_i(n) \triangle G_i) < \varepsilon/2mk_m, 1 < i < m \). Define \( f(x) = k_i, x \in E_i(n), = 0, x \in X \setminus \bigcup_{i=1}^m E_i(n) \). Then

\[
\int_X |f - g| \, d\mu = \int_{\bigcup_{i=1}^m G_i} |f - g| \, d\mu + \int_{X \setminus \bigcup_{i=1}^m G_i} |f - g| \, d\mu
\]

\[
= \sum_{i=1}^m \left( \int_{G_i} |f - g| \, d\mu \right) + \int_{X \setminus \bigcup_{i=1}^m G_i} |f - g| \, d\mu
\]

\[
< \sum_{i=1}^m \int_{G_i \cap E_i(n)} |f - g| \, d\mu + \int_{G_i \setminus E_i(n)} |f - g| \, d\mu + 2 \int_{X \setminus \bigcup_{i=1}^m G_i} g \, d\mu,
\]

since \( f(x) < g(x) \) if \( x \in X \setminus \bigcup_{i=1}^m G_i \). Thus by (2),

\[
\int_X |f - g| \, d\mu < \sum_{i=1}^m \int_{G_i \setminus E_i(n)} |f - g| \, d\mu + \frac{\varepsilon}{2} < \sum_{i=1}^m \int_{G_i \setminus E_i(n)} |f - k_i| \, d\mu + \frac{\varepsilon}{2}
\]

\[
< \sum_{i=1}^m k_i \mu(G_i \triangle E_i(n)) + \frac{\varepsilon}{2} < \sum_{i=1}^m \frac{k_i \varepsilon}{2mk_m} + \frac{\varepsilon}{2} = \varepsilon.
\]

The result below is analogous to Theorem 3.

Theorem 6. Let \( T: X \to X \) be an automorphism which admits a cyclic a.p.t. \( I \) with respect to a sequence of partitions \( \xi(n) \to \varepsilon_X, \xi(n) = \{C_i(n): 1 < i < q(n)\} \), with speed \( g(n) = o(1/n^2) \). Let \( f: X \to \mathbb{N} \) be integrable, and suppose that for each \( C_i(n) \in \xi(n), 1 < i < q(n) \), there exists \( k_i(n) \in \mathbb{N} \) such that

\[
\mu \left\{ x \in C_i(n): f(x) = k_i(n) \right\} > (1 - \varepsilon(n)) \mu(C_i(n)),
\]

(3)

where \( \varepsilon(n) \) is a sequence of real numbers with \( \varepsilon(n) = o(1/q(n)) \). Then \( T_f \) admits a simple approximation.

Proof. As in Theorem 3, we can show that there exists a sequence of partitions \( \xi(n) = \{C_i(n): 1 < i < q(n)\} \) with \( \xi(n) \to \varepsilon_X \) and \( \mu(C_i(n) \setminus C_i'(n)) < \frac{1}{2} g(q(n)) \) such that \( T \) admits a simple approximation with respect to the sequence \( \{\xi(n)\} \).

We show that \( T \) and \( \{\xi(n)\} \) satisfy the conditions of Theorem 4, from which it will follow that \( T_f \) admits a simple approximation.

Let \( B_i(n) = f^{-1}(k_i(n)) \). Then by (3),
\[
\mu\left(C'_{i}(n) \cap B_{j}(n)\right) = \mu\left(C_{i}(n) \cap B_{j}(n)\right) - \mu\left((C'_{i}(n) \setminus C_{i}(n)) \cap B_{j}(n)\right)
\]
\[
> (1 - \epsilon(n)) \mu(C_{i}(n)) - \frac{1}{2} g(q(n))
\]
\[
> (1 - \epsilon(n)) \mu(C'_{i}(n)) - \frac{1}{2} g(q(n)).
\]

Put \( \delta(n) = \epsilon(n) + g(q(n))/2\mu(C_{i}(n)) \); then \( \delta(n) = o(1/q(n)) \). Hence,
\[
\mu\{x \in C'_{i}(n) : f(x) = k_{j}(n)\} = \mu(C'_{i}(n) \cap B_{j}(n)) > (1 - \delta(n)) \mu(C_{i}(n)),
\]
and so by Theorem 4, \( T_{j} \) admits a simple approximation.

The above results give conditions for a special automorphism to admit a simple approximation. We now give conditions for a special automorphism to admit a cyclic a.p.t.I with a preassigned speed.

**Definition 6.** Let \( T : X \to X \) be an automorphism. Then \( T \) admits a cyclic a.p.t.I with speed \( g(x) \) if \( T \) admits a cyclic a.p.t.I with speed \( g(n) \), where \( g \) is a decreasing function from \( \mathbb{R}^{+} \) to \( \mathbb{R}^{+} \) with \( g(x) \to 0 \) as \( x \to \infty \).

**Theorem 7.** Let \( T : X \to X \) be an automorphism which admits a cyclic a.p.t.I with respect to a sequence of partitions \( \xi(n) \to \varepsilon_{x} \), \( \xi(n) = \{C_{i}(n) : 1 < i < q(n)\} \) with speed \( g(x) = o(1/x^{k}) \). Let \( f : X \to \mathbb{N} \) be integrable with \( f(C_{i}(n)) = k_{j}(n) \in \mathbb{N}, 1 < i < q(n), n > 1 \). Then \( T_{j} \) admits a cyclic a.p.t.I with speed \( o(1/n^{k}) \).

**Proof.** Let \( \xi^{f}(n) = \{C_{i}(n) : 1 < i < q(n), 0 < j < k_{i}(n)\}, \) where \( C_{i,0}(n) = C_{i}(n) \in \xi(n) \). For ease of notation put
\[
C_{i,k_{i}+1}(n) = C_{i+1,0}(n) \text{ and } C_{q(n)+1,k_{q(n)}+1}(n) = C_{1,0}(n).
\]
Then
\[
\sum_{i=1}^{q(n)} \sum_{j=0}^{k_{i}(n)} \mu'(T_{j}C_{i}(n) \triangle C_{i+1}(n))
\]
\[
= \sum_{i=1}^{q(n)} \mu'(T_{j}C_{i,k_{i}(n)}(n) \triangle C_{i+1,0}(n))
\]
\[
= \left(1 + \int f d\mu\right)^{-1} \sum_{i=1}^{q(n)} \mu(T_{j}C_{i}(n) \triangle C_{i+1}(n))
\]
\[
< \left(1 + \int f d\mu\right)^{-1} g(q(n)).
\]

Let \( p(n) \) be the number of elements in \( \xi^{f}(n) \). Then
\[
p(n) = q(n) + \sum_{i=1}^{q(n)} k_{i}(n) = q(n) \left(1 + \frac{1}{q(n)} \sum_{i=1}^{q(n)} k_{i}(n)\right)
\]
\[
< q(n) \left(1 + 2\mu(C_{1}(n)) \sum_{i=1}^{q(n)} k_{i}(n)\right),
\]
for \( n \) sufficiently large, since \( q(n) \mu(C_{1}(n)) \to 1 \) and so \( q(n) \mu(C_{1}(n)) > \frac{1}{2} \) for \( n \) sufficiently large.
We may assume without loss of generality that (4) holds for all $n$. Hence,
\[
p(n) < q(n) \left( 1 + 2\mu(C_1(n)) \sum_{i=1}^{q(n)} k_i(n) \right) < q(n) \left( 1 + 2 \int f d\mu \right).
\]  
(5)

Now put $G(n) = g(n \cdot (1 + 2\int f d\mu)^{-1}) \cdot (1 + 2\int f d\mu)^{-1}$. Then,
\[
\sum_{i=1}^{q(n)} \sum_{j=0}^{k_i(n)} \mu'(T_j C_i(n) \triangle C_{i,j+1}(n)) \leq \left( 1 + \int f d\mu \right)^{-1} g(q(n)) < G(p(n))
\]
by (5).

The conclusion of the theorem is now apparent.

By exactly the same reasoning we also have the following.

**Theorem 8.** Let $T: X \to X$ be an automorphism which admits a cyclic a.p.t.I with respect to a sequence of partitions $\{\xi(n)\}$, $\xi(n) = \{C_i(n): 1 \leq i \leq q(n)\}$ with speed $g(x) = \theta / x$, $\theta \in \mathbb{R}^+$. Let $f: X \to \mathbb{N}$ be integrable with $f(C_i(n)) = k_i(n) \in \mathbb{N}$, $1 \leq i \leq q(n)$, $n \geq 1$. Then $T_f$ admits a cyclic a.p.t.I with speed $G(n) = 2\theta / n$.

**Proof.** As for Theorem 7, noticing that
\[
\left( 1 + \int f d\mu \right)^{-1} g(q(n)) < \theta \left( 1 + 2 \int f d\mu \right) / p(n) \left( 1 + \int f d\mu \right) < 2\theta / p(n).
\]

It is known [1] that if $T$ admits a cyclic a.p.t.I with speed $\theta / n$, where $\theta < 1$, then $T$ has simple spectrum.

We do not know if, in general, a sufficient condition for $T_f$ to have simple spectrum is that $T$ has simple spectrum.

**Bibliography**


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