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α-RECOGNIZABLE SEMIGROUPS

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ABSTRACT. Semigroups in which every principal congruence is of finite index are studied in this paper.

1. Introduction. In the theory of automata, a subset A of Σ^*, the free monoid on the alphabet Σ, is said to be recognizable iff there exists a finite monoid M, an epimorphism ϕ: $\Sigma^* \to M$ and a subset $B \subset M$ such that $A = B\phi^{-1}$. Equivalently, a subset A of Σ^* is recognizable iff the principal congruence P_A is of finite index in Σ^*, or more briefly, finite in Σ^*. Since P_A is finite in Σ^* iff the principal right congruence R_A is finite in Σ^*, a subset A of Σ^* is recognizable iff R_A is finite in Σ^*. The principal congruence P_A is defined as follows: $x \equiv y[P_A]$ iff $A..x = A..y$, where for $w \in \Sigma^*$,

$$A..w = \{(u, v) \in \Sigma^* \times \Sigma^* | uwv \in A\}.$$

The principal right congruence is given by $x \equiv y[R_A]$ iff $x^{-1}A = y^{-1}A$, i.e. $\{u \in \Sigma^* | xu \in A\} = \{u \in \Sigma^* | yu \in A\}$.

Each of these equivalent definitions can be stated for arbitrary monoids [1] and for semigroups in general. The notion of a recognizable set in a semigroup is defined as follows: if S is a semigroup, S^1 shall denote S if S has an identity, otherwise $S^1 = S \cup \{1\}$ where the adjoined 1 is the identity. For a subset A of S, the principal congruence P_A of A in S is defined by $x \equiv y[P_A]$ iff $A..x = A..y$, where for $w \in S$,

$$A..w = \{(u, v) \in S^1 \times S^1 | uwv \in A\}.$$

P_A is thus the coarsest congruence saturating A and the canonical epimorphism $S \to S/P_A$ factors uniquely through any epimorphism whose kernel saturates A. A subset A of S is then said to be recognizable iff P_A is finite in S.

It is apparent that, while the equivalent forms of the definition of recognizable set in Σ^* can be formulated for any semigroup and remain equivalent there, and the properties of the set of recognizable subsets, such as the closure under Boolean operations and inverse image, hold in the more general setting, there are many differences between the case of the free monoid and that of

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monoids or semigroups in general. For example, in Σ^* every singleton set, and hence every finite set, is recognizable. However, any disjunctive [4] subset of an infinite semigroup is not recognizable, and thus every infinite subdirectly irreducible semigroup contains a singleton which is not recognizable, while in an infinite group, no singleton is recognizable. It would be interesting to determine the properties of semigroups for which every singleton is recognizable.

Our purpose in this paper is to study semigroups for which every subset is recognizable. Such a semigroup shall be called α-recognizable. Of course, every finite semigroup is α-recognizable, and α-recognizability is a strong finiteness condition. However, the class of infinite α-recognizable semigroups is not empty and our investigations are aimed at determining properties of α-recognizable semigroups modulo finite semigroups.

2. α-recognizable semigroups.

2.1 THEOREM. *Each homomorphic image of an α-recognizable semigroup is α-recognizable.*

PROOF. Let S be α-recognizable and ϕ: $S \to T$ an epimorphism. Let $B \subset T$ and $A = B\phi^{-1}$. We obtain the following commutative diagram:

$$\begin{tikzcd}
S \arrow[r, "\phi"] \arrow[d, "\pi"'] & T \arrow[d, "\pi"] \arrow[dl, dashed, "\pi_1"'] \\
S/P_A \arrow[r, dashed, "\pi_2"'] & T/P_B
\end{tikzcd}$$

since $A\pi = B\pi_1$ implies $A = A\pi\pi^{-1} = B\pi_1\pi_1^{-1}\phi^{-1}$ and so $B = A\phi = (B\pi_1)\pi_1^{-1}$. Since A is recognizable, S/P_A is finite and so T/P_B is finite, whence B is recognizable. $\square$

Since, as we noted earlier, a subdirectly irreducible α-recognizable semigroup must be finite, every α-recognizable semigroup is a subdirect product of finite semigroups.

For any congruence ϕ on a semigroup S, the index of ϕ in S shall be denoted by $[S : \phi]$.

2.2 LEMMA. *Let S be a semigroup and T be a subsemigroup of S. Let $A \subset T$. Then $[T : P_A] \leqslant [S : P_A]$.*

2.3 COROLLARY. *Every subsemigroup of an α-recognizable semigroup is α-recognizable.*

Thus, in particular, every subgroup of an α-recognizable semigroup is finite. In fact, the subgroups of an α-recognizable semigroup have bounded order.

2.4 THEOREM. *For each α-recognizable semigroup S there exists $k_S \in N$ such*

that if G is a subgroup of S, then $|G| \leqslant k_S$.

PROOF. Suppose not. Let $\{G_i\}$ be a sequence of subgroups of S with $|G_i| = n_i$ and $n_i < n_{i+1}$ for all $i \geqslant 1$. Let e_i be the identity of G_i and put $A = \{e_i | i \in N\}$. For any $x, y \in G_i$, if $x \neq y$ then $x \not\equiv y[P_A]$ since $(1, x^{-1}) \in A. . x$ but $(1, x^{-1}) \notin A. .y$. Thus $[S : P_A] \geqslant n_i$ for all i, whence P_A is not finite, a contradiction. □

That an α-recognizable semigroup has subgroups is one of the consequences of the following result.

2.5 THEOREM. *For each α-recognizable semigroup S there exist $m, n \in N$ such that S satisfies the identity $x^{m+n} = x^m$.*

PROOF. Let S be an α-recognizable semigroup. We prove first that S is periodic. If it were not, then it would contain an aperiodic element a. Let $A = \{a^k | k = 2^t, t \in N\}$. Then the elements of the form a^m, $m = 2^t + 1$, are pairwise inequivalent modulo P_A. For if $j < k$ with $m = 2^j - 1$ and $t = 2^k - 1$, then $(1, a^t) \in A. .a^{t+2}$ but $(1, a) \notin A. .a^{m+2}$. Thus P_A is infinite, a contradiction. Thus S is periodic.

Now let $E = E(S)$ denote the set of idempotents of S. Then S/P_E is finite. Let $m = |S/P_E|$. For each $x \in S$ there exists $k \leqslant m$ such that $[a^k]$ is an idempotent of S/P_E, whence $a^k \in E$. Thus for each $x \in S$, x^m belongs to a subgroup of S. Since each subgroup of S has order not exceeding k_S, $(x^m)^{(k_S)!}$ is the identity of the subgroup containing x^m.

Finally, for $n = m(k_S!)$, we obtain $x^{m+n} = x^m$ for any $x \in S$. □

2.6 THEOREM. *An α-recognizable semigroup has no infinite chains of idempotents.*

PROOF. Suppose that $\{e_i\}$ is a sequence of idempotents such that if $i < j$, then $e_i < e_j$. Let $A = \{e_{2n+1} | n \in N\}$. Then for $n \in N$, $e_{2n} \not\equiv e_{2n+2k}[P_A]$ for any $k \in N$, since $(1, e_{2n+1}) \in A. .e_{2n+2k}$ but $(1, e_{2n+1}) \notin A. .e_{2n}$. Thus P_A is infinite, a contradiction. A similar proof can be given to show that there are no infinite strictly decreasing sequences of idempotents. □

Thus an α-recognizable semigroup has minimal idempotents.

2.7 COROLLARY. *If S is an α-recognizable semigroup and G is a group homomorphic image of S, then $|G| \leqslant k_S$.*

PROOF. The well-known result (see, for example, [**2**, p. 36]) that every group homomorphic image of a finite semigroup is the image of a subgroup of the semigroup obviously holds for periodic semigroups with a minimal idempotent. □

A similar result holds for cyclic semigroup homomorphic images of an α-recognizable semigroup, with the bound $b_S = \sup\{|\langle x \rangle| \mid x \in S\}$.

2.8 LEMMA. *An α-recognizable semilattice is finite.*

PROOF. Suppose $A = \{e_1, e_2, \ldots\}$ is a countably infinite set of

incomparable elements of an α-recognizable semilattice S. Then for each i, $j \in N$, with $i \neq j$, we have $e_i \not\equiv e_j[P_A]$. For (e_i, e_i) belongs to $A..e_i$ but not $A..e_j$. Thus P_A is infinite, a contradiction, and so all sets of incomparable elements are finite. Since all chains of idempotents must be finite, S must be finite. □

2.9 COROLLARY. *A commutative α-recognizable semigroup has only finitely many idempotents.* □

2.10 COROLLARY. *Any semilattice decomposition of an α-recognizable semigroup has only finitely many components.* □

We wish to thank the referee for the following result.

2.11 THEOREM. *An α-recognizable inverse semigroup is finite.*

PROOF. The set of idempotents of an inverse semigroup S forms a semilattice. If S is α-recognizable, then $E(S)$ is finite by (2.8). Each L-class and each R-class of S contains an idempotent, so there are only finitely many L and R classes, hence only finitely many H classes. Each D-class contains an H-class which is a subgroup of S and thus finite, whence all H-classes contained in that D-class are finite. Thus S is a finite union of finite sets, hence finite. □

For our first examples of infinite α-recognizable semigroups we observe that any left zero semigroup (right zero semigroup) is α-recognizable. Not only are they α-recognizable, but the principal congruences in such semigroups have index not greater than two. This suggests another bound which might be associated with an α-recognizable semigroup S. Let C_S denote the least natural number such that for all $A \subset S$, $[S : P_A] \leqslant C_S$, if such a number exists. T. Tamura has shown [5] that a semigroup S is singular (i.e. a left zero semigroup or a right zero semigroup) or else a zero semigroup iff $C_S \leqslant 2$. Thus zero semigroups are α-recognizable as well. More generally, we have the following result. Define congruences [4] e_R and e_L on a semigroup S by $x \equiv y[e_R]$ iff $xg = yg$ for all $g \in S$ and $x \equiv y[e_L]$ iff $gx = gy$ for all $g \in S$. Let $\pi = e_R \cap e_L$.

2.12 THEOREM. *If π is finite then S is α-recognizable and $C_S \leqslant [S : \pi]$.*

PROOF. Let $A \subset S$. If there are $x, y \in S$ such that $x \not\equiv y[P_A]$, then there are $u, v \in S$ such that $uxv \in A$ and $uxv \notin A$ or vice-versa. Thus $uxv \neq uyv$ whence $ux \neq uy$ and so $x \not\equiv y[\pi]$. □

Since π is finite for any inflation of a finite semigroup, we see that any inflation of a finite semigroup is α-recognizable. In fact, any inflation T of any α-recognizable semigroup S is α-recognizable and $C_T = C_S$ except when $S = \{0\}$ in which case $C_T = 2$, $C_S = 1$.

However, π is not of finite index for all α-recognizable semigroups as Example 2.18 shows.

In addition to the above finiteness results, one might also suspect that α-recognizable semigroups have the acc or the dcc on ideals, either one- or

two-sided, but the examples offered above show that in general this is not the case, even for commutative α-recognizable semigroups.

2.13 THEOREM. *The direct product of two α-recognizable monoids is α-recognizable iff at least one is finite.*

PROOF. Let M_1 and M_2 be α-recognizable monoids. By Proposition 12.2 of [1], a subset $A \subset M_1 \times M_2$ is recognizable iff A is the finite union of sets of the form $B \times C$ where B is a recognizable subset of M_1 and C is a recognizable subset of M_2. Thus if at least one of M_1 and M_2 is finite, $M_1 \times M_2$ is α-recognizable.

Conversely, if both M_1 and M_2 are infinite monoids, choose a sequence $\{a_i\}$ of distinct elements of M_1 and a sequence of distinct elements $\{b_i\}$ of M_2. Then the subset $A = \{(a_i, b_i) | i \in N\}$ is not the finite union of products of recognizable subsets of M_1 and M_2, respectively, whence A is not recognizable. $\square$

It is interesting to note that this result does not hold for α-recognizable semigroups in general. For example, the product of any two left zero semigroups is again a left zero semigroup, hence α-recognizable. Moreover, if S is a left zero semigroup, then S^1 is α-recognizable and so if S is infinite, $S^1 \times S^1$ is not α-recognizable while $S \times S$ is α-recognizable.

We do have the following result, which suggests that the preceding example is quite pathological.

2.14 THEOREM. *A rectangular band $S = L \times R$ is α-recognizable iff at least one of the factors is finite.*

PROOF. First, suppose that both the left zero semigroup L and the right zero semigroup R are infinite. Let $\{a_i\}$ and $\{b_i\}$ be sequences of distinct elements from L and R, respectively, and let $A = \{(a_i, b_i) | i \in N\}$. For each $i \in N$,

$$A..(a_i, b_i) = \{((a_j, u), (v, b_j)) | u, v \in S, j \in N\}$$
$$\cup (1 \times \{(u, b_i) | u \in S\}) \cup (\{(a_i, v) | v \in S\} \times 1) \cup \{(1, 1)\}.$$

Thus for $i \neq j$, $A..(a_i, b_i) \neq A..(a_j, b_j)$ and so A is not recognizable.

Conversely, consider the monoid $L^1 \times R^1$. Since both L^1 and R^1 are α-recognizable, $L^1 \times R^1$ is α-recognizable iff at least one of L^1 or R^1 is finite. If $L^1 \times R^1$ is α-recognizable, then so is its subsemigroup $L \times R$. $\square$

There is an example due to B. M. Schein [6] of an infinite subdirectly irreducible band. This semigroup cannot be α-recognizable since it is infinite and subdirectly irreducible. However it does satisfy an identity of the form $x^{m+n} = x^m$, namely $x^2 = x$. All of its subgroups and group homomorphic images are finite of bounded order, in fact they are singletons. There are no chains of idempotents of length greater than two, and, finally, it is a finite semilattice of α-recognizable semigroups (a right zero semigroup and a left zero semigroup) whence we see that a finite semilattice of α-recognizable semigroups need not be α-recognizable.

Even though a finite semilattice of α-recognizable semigroups need not be α-recognizable, it might still prove useful to consider semilattice decompositions of an α-recognizable semigroup. In particular, if one considers a commutative α-recognizable semigroup S, the maximum semilattice decomposition of S presents S as a finite semilattice of periodic archimedean homogroups each with finite group kernel.

If one defines a power-absorbing set in a semigroup S to be a subset A of S such that for each $t \in S$ there exists $n_t \in N$ such that $t^n \in A$ for all $n \geqslant n_t$, then, for example, S is archimedean iff every ideal is power-absorbing.

2.15 LEMMA. *Let S be a semigroup with a power-absorbing recognizable ideal I. Then for some $n \in N$, $S^n \subset I$.*

PROOF. Since I is an ideal, I is a class of P_I. Thus S/P_I is a finite nil semigroup with zero I, and so for $n = |S/P_I|$, $S^n \subset I$. □

2.16 THEOREM. *Let S be an α-recognizable semigroup with $E(S) = \{e\}$. Then for some $n \in N$, $S^n = G$, the group ideal of S.*

PROOF. S is periodic and so e is central, whence the group $G = eSe$ is an ideal of S. Thus for $n = |S/P_G|$, $S^n = G$. □

Thus, in particular, a commutative semilattice-indecomposable α-recognizable semigroup S with finite group kernel G satisfies $S^n = G$ for some $n \in N$.

Up to this stage, the only examples of commutative semilattice-indecomposable α-recognizable semigroups have been inflations of finite commutative semigroups. There exists an infinite semilattice-indecomposable commutative α-recognizable semigroup which is not an inflation of a finite semigroup.

We observe that if we are to find such a semigroup, we must examine semigroups which are not finitely generated, since any commutative periodic finitely generated semigroup is finite. It is perhaps a little surprising that this is true for noncommutative α-recognizable semigroups as well. We wish to thank J. Sakarovitch for pointing this out to us.

2.17 THEOREM [7]. *Let S be a finitely generated infinite semigroup. Then there exists a subset A of S for which P_A is infinite.*

Thus a finitely generated α-recognizable semigroup is finite.

2.18 EXAMPLE. Let $H = \{x_1, x_2, \ldots\}$, $L = \{a_1, a_2, \ldots\}$ and put $S = H \cup L \cup \{0\}$. Define multiplication on S as follows:

(i) $S^3 = \{0\}$;

(ii) $x_1x_j = x_jx_1 = a_j$ for all $j \in N$;

(iii) $x_ix_j = x_jx_i = a_1$ for all $i \neq 1, j \neq 1$;

(iv) $a_ia_j = 0$ for all $i, j \in N$;

(v) $0x = x0 = 0$ for all $x \in S$.

Then S is a commutative semigroup. Moreover, $E(S) = \{0\}$ and so S is semilattice-indecomposable. Since $S^2 = L \cup \{0\}$, S is not an inflation of a

finite commutative semigroup. Finally, for $A \subset S$, $\{A..x | x \in S\}$ is finite and so S is α-recognizable.

In fact, in this example, $[S : P_A]$ is bounded for $A \subset S$, while (ii) implies that π is infinite.

Finally, it is interesting to observe that an α-recognizable semilattice-indecomposable semigroup is (modulo the group ideal) partially-ordered by division. That is to say, if e is the idempotent of S and G is the group ideal, then for $x, y \in (S \setminus G) \cup \{e\}$, $x \leqslant y$ if $x \in yS^1$. This ordering is compatible with the multiplication and is useful for geometrically portraying semilattice-indecomposable α-recognizable semigroups.

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