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tan x IS ERGODIC

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ABSTRACT. It is proved that the transformation $x \mapsto \tan x$ is ergodic on the real line with respect to Lebesgue measure.

Let S be a transformation of the measure space $(\mathbf{R}, \beta, \lambda)$. Here, $\mathbf{R}$ denotes the real axis, β the σ-field of all Lebesgue measurable subsets of $\mathbf{R}$, and λ Lebesgue measure. The transformation S is called ergodic (with respect to λ) if $S^{-1}E = E$ implies $\lambda(E) = 0$ or $\lambda(\mathbf{R} \setminus E) = 0$. J. H. B. Kemperman [1] has shown ergodicity for a wide class of transformations S. His results include the transformation $Sx = a \tan x + b$ for various choices of the parameters a and b. His methods left open the simple case $Sx = \tan x$.

The purpose of this note is to prove:

THEOREM. *The transformation* $x \mapsto \tan x$ *is ergodic.*

PROOF. We first note that after a removal of a countable set of points the pair $(\mathbf{R}, S)$ is a fibered system in the sense of Schweiger [2].

The fibres are given by $B(k) =]k\pi - \pi/2, k\pi + \pi/2[$, $k = 0, \pm 1, \pm 2, \ldots$.

As usual we define cylinders by $B(k_1, \ldots, k_n) := B(k_1) \cap S^{-1}B(k_2, \ldots, k_n)$.

With each cylinder we associate the map $V(k_1, \ldots, k_n)\colon \mathbf{R} \to B(k_1, \ldots, k_n)$ defined as the inverse map of the restriction of S^n to the set $B(k_1, \ldots, k_n)$. Note that $V(k)x = \arctan x + k\pi$. The Jacobian $\omega(k_1, \ldots, k_n)$ is given by $\lambda(B(k_1, \ldots, k_n) \cap S^{-n}E) = \int_E \omega(k_1, \ldots, k_n)$.

Therefore $\omega(k)x = 1/(1 + x^2)$ is independent of the digit k.

Now assume that $S^{-1}E = E$ and $\lambda(E) > 0$. It is sufficient to prove: There is a constant $Q > 0$ such that

$$\lambda(B(k_1, \ldots, k_n) \cap E) \geqslant Q\lambda(B(k_1, \ldots, k_n))$$

for any cylinder with $k_n \neq 0$.

Now we calculate

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$$\frac{\lambda(B(k_1,\ldots,k_n)\cap E)}{\lambda(B(k_1,\ldots,k_n))}=\frac{\lambda(B(k_1,\ldots,k_n)\cap S^{-n}E)}{\lambda(B(k_1,\ldots,k_n))}$$
$$=\frac{\int_E\omega(k_1,\ldots,k_n)}{\int_{\mathbf{R}}\omega(k_1,\ldots,k_n)}=\frac{\Sigma_b\int_{B(b)\cap E}\omega(k_1,\ldots,k_n)}{\Sigma_b\int_{B(b)}\omega(k_1,\ldots,k_n)}$$
$$>\frac{\Sigma_b(\inf_{x\in B(b)}\omega(k_1,\ldots,k_n)(x))\lambda(B(b)\cap E)}{\Sigma_b(\sup_{x\in B(b)}\omega(k_1,\ldots,k_n)(x)\lambda(B(b))}.$$

Since $\omega(b)$ does not depend on b, one calculates that

$$\frac{\lambda(B(b)\cap E)}{\lambda(B(b))}=\frac{\lambda(B(b)\cap S^{-1}E)}{\lambda(B(b))}=Q_1,\qquad Q_1\text{ constant.}$$

Therefore it is sufficient to show the existence of a constant Q_2 such that

$$\inf_{x\in B(b)}\omega(k_1,\ldots,k_n)(x)\geqslant Q_2\sup_{x\in B(b)}\omega(k_1,\ldots,k_n)(x)$$

Now

$$\left|\log\frac{\omega(k_1,\ldots,k_n)(x)}{\omega(k_1,\ldots,k_n)(y)}\right|$$
$$\leqslant\sum_{i=1}^{n}\left|\log\big(\omega(k_i)(V(k_{i+1},\ldots,k_n)x)\big)-\log\big(\omega(k_i)(V(k_{i+1},\ldots,k_n)y)\big)\right|$$

(formally $V(k_{i+1},\ldots,k_n)t=t$ for $i=n$)

$$\leqslant\sum_{i=1}^{n}\sup\left|\frac{\omega(k_i)'(\xi)}{\omega(k_i)(\xi)}\right|\,|V(k_{i+1},\ldots,k_n)x-V(k_{i+1},\ldots,k_n)y|.$$

Now

$$\left|\frac{\omega(k_i)'(\xi)}{\omega(k_i)(\xi)}\right|=\left|\frac{2\xi}{1+\xi^2}\right|\leqslant 1.$$

If $x\in B(b)$ and $y\in B(b)$, then $V(k_{i+1},\ldots,k_n)x$ and $V(k_{i+1},\ldots,k_n)y$ both belong to the interval $B(k_{i+1},\ldots,k_n,b)$ resp. $B(b)$ (for the case $i=n$). Therefore

$$\sup_{x,y\in B(b)}\left|\log\frac{\omega(k_1,\ldots,k_n)(x)}{\omega(k_1,\ldots,k_n)(y)}\right|\leqslant\sum_{i=1}^{n-1}\lambda(B(k_{i+1},\ldots,k_n,b))+\lambda(B(b)).$$

Since $\omega(k)(x)=(1+x^2)^{-1}$ is independent of k and symmetric to the origin one verifies easily that

$$\lambda(B(k_{i+1},\ldots,k_{n-1},k_n,b))\leqslant\lambda(B(0,\ldots,0,k_n,b))$$

which is obvious by the symmetry of the problem. Since $k_n\neq 0$ the $n-2$ cylinders $B(0,\ldots,0,k_n,b)$ are pairwise disjoint subsets of $B(0)$ (for different numbers of zeroes), we obtain

$$\sup_{x,y\in B(b)}\left|\log\frac{\omega(k_1,\ldots,k_n)(x)}{\omega(k_1,\ldots,k_n)(y)}\right| < \lambda(B(0)) + \lambda(B(k_n)) + \lambda(B(b)) = 3\pi.$$

Hence $Q_2 = e^{-3\pi}$ is a suitable constant.

References

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