

$\tan x$ IS ERGODIC

F. SCHWEIGER

ABSTRACT. It is proved that the transformation $x \mapsto \tan x$ is ergodic on the real line with respect to Lebesgue measure.

Let S be a transformation of the measure space $(\mathbf{R}, \beta, \lambda)$. Here, \mathbf{R} denotes the real axis, β the σ -field of all Lebesgue measurable subsets of \mathbf{R} , and λ Lebesgue measure. The transformation S is called ergodic (with respect to λ) if $S^{-1}E = E$ implies $\lambda(E) = 0$ or $\lambda(\mathbf{R} \setminus E) = 0$. J. H. B. Kemperman [1] has shown ergodicity for a wide class of transformations S . His results include the transformation $Sx = a \tan x + b$ for various choices of the parameters a and b . His methods left open the simple case $Sx = \tan x$.

The purpose of this note is to prove:

THEOREM. *The transformation $x \mapsto \tan x$ is ergodic.*

PROOF. We first note that after a removal of a countable set of points the pair (\mathbf{R}, S) is a fibered system in the sense of Schweiger [2].

The fibres are given by $B(k) =]k\pi - \pi/2, k\pi + \pi/2[$, $k = 0, \pm 1, \pm 2, \dots$

As usual we define cylinders by $B(k_1, \dots, k_n) := B(k_1) \cap S^{-1}B(k_2, \dots, k_n)$.

With each cylinder we associate the map $V(k_1, \dots, k_n): \mathbf{R} \rightarrow B(k_1, \dots, k_n)$ defined as the inverse map of the restriction of S^n to the set $B(k_1, \dots, k_n)$. Note that $V(k)x = \arctan x + k\pi$. The Jacobian $\omega(k_1, \dots, k_n)$ is given by $\lambda(B(k_1, \dots, k_n) \cap S^{-n}E) = \int_E \omega(k_1, \dots, k_n)$.

Therefore $\omega(k)x = 1/(1+x^2)$ is independent of the digit k .

Now assume that $S^{-1}E = E$ and $\lambda(E) > 0$. It is sufficient to prove: There is a constant $Q > 0$ such that

$$\lambda(B(k_1, \dots, k_n) \cap E) \geq Q\lambda(B(k_1, \dots, k_n))$$

for any cylinder with $k_n \neq 0$.

Now we calculate

Received by the editors June 14, 1977.

AMS (MOS) subject classifications (1970). Primary 28A65; Secondary 10K10.

Key words and phrases. Ergodicity, transformations of the real line into itself, dynamical systems.

© American Mathematical Society 1978

$$\begin{aligned} \frac{\lambda(B(k_1, \dots, k_n) \cap E)}{\lambda(B(k_1, \dots, k_n))} &= \frac{\lambda(B(k_1, \dots, k_n) \cap S^{-n}E)}{\lambda(B(k_1, \dots, k_n))} \\ &= \frac{\int_E \omega(k_1, \dots, k_n)}{\int_{\mathbb{R}} \omega(k_1, \dots, k_n)} = \frac{\sum_b \int_{B(b) \cap E} \omega(k_1, \dots, k_n)}{\sum_b \int_{B(b)} \omega(k_1, \dots, k_n)} \\ &> \frac{\sum_b (\inf_{x \in B(b)} \omega(k_1, \dots, k_n)(x)) \lambda(B(b) \cap E)}{\sum_b (\sup_{x \in B(b)} \omega(k_1, \dots, k_n)(x)) \lambda(B(b))}. \end{aligned}$$

Since $\omega(b)$ does not depend on b , one calculates that

$$\frac{\lambda(B(b) \cap E)}{\lambda(B(b))} = \frac{\lambda(B(b) \cap S^{-1}E)}{\lambda(B(b))} = Q_1, \quad Q_1 \text{ constant.}$$

Therefore it is sufficient to show the existence of a constant Q_2 such that

$$\inf_{x \in B(b)} \omega(k_1, \dots, k_n)(x) \geq Q_2 \sup_{x \in B(b)} \omega(k_1, \dots, k_n)(x)$$

Now

$$\begin{aligned} &\left| \log \frac{\omega(k_1, \dots, k_n)(x)}{\omega(k_1, \dots, k_n)(y)} \right| \\ &\leq \sum_{i=1}^n \left| \log(\omega(k_i)(V(k_{i+1}, \dots, k_n)x)) - \log(\omega(k_i)(V(k_{i+1}, \dots, k_n)y)) \right| \\ &\quad \text{(formally } V(k_{i+1}, \dots, k_n)t = t \text{ for } i = n) \\ &\leq \sum_{i=1}^n \sup \left| \frac{\omega(k_i)'(\xi)}{\omega(k_i)(\xi)} \right| |V(k_{i+1}, \dots, k_n)x - V(k_{i+1}, \dots, k_n)y|. \end{aligned}$$

Now

$$\left| \frac{\omega(k_i)'(\xi)}{\omega(k_i)(\xi)} \right| = \left| \frac{2\xi}{1 + \xi^2} \right| \leq 1.$$

If $x \in B(b)$ and $y \in B(b)$, then $V(k_{i+1}, \dots, k_n)x$ and $V(k_{i+1}, \dots, k_n)y$ both belong to the interval $B(k_{i+1}, \dots, k_n, b)$ resp. $B(b)$ (for the case $i = n$). Therefore

$$\sup_{x,y \in B(b)} \left| \log \frac{\omega(k_1, \dots, k_n)(x)}{\omega(k_1, \dots, k_n)(y)} \right| \leq \sum_{i=1}^{n-1} \lambda(B(k_{i+1}, \dots, k_n, b)) + \lambda(B(b)).$$

Since $\omega(k)(x) = (1 + x^2)^{-1}$ is independent of k and symmetric to the origin one verifies easily that

$$\lambda(B(k_{i+1}, \dots, k_{n-1}, k_n, b)) \leq \lambda(B(0, \dots, 0, k_n, b))$$

which is obvious by the symmetry of the problem. Since $k_n \neq 0$ the $n - 2$ cylinders $B(0, \dots, 0, k_n, b)$ are pairwise disjoint subsets of $B(0)$ (for different numbers of zeroes), we obtain

$$\sup_{x, y \in B(b)} \left| \log \frac{\omega(k_1, \dots, k_n)(x)}{\omega(k_1, \dots, k_n)(y)} \right| < \lambda(B(0)) + \lambda(B(k_n)) + \lambda(B(b)) = 3\pi.$$

Hence $Q_2 = e^{-3\pi}$ is a suitable constant.

REFERENCES

1. J. H. B. Kemperman, *The ergodic behavior of a class of real transformations*, Stochastic Processes and Related Topics (Proc. Summer Res. Inst. on Statist. Inference for Stochastic Processes, Indiana Univ., Bloomington, Indiana, 1974), Academic Press, New York, 1975, pp. 249–258.
2. F. Schweiger, *Some remarks on ergodicity and invariant measures*, Michigan Math. J. **22** (1975), 181–187.

INSTITUT FÜR MATHEMATIK DER UNIVERSITÄT SALZBURG, PETERSBRUNNSTRASSE 19, A-5020 SALZBURG, AUSTRIA