INNER AMENABLENESS AND CONJUGATION OPERATORS

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Abstract. It is shown that an infinite discrete group $G$ is inner amenable if and only if the C*-algebra generated by the unitaries on $l^2(G)$ corresponding to conjugation by $s (s \in G)$ does not contain the projection on the point-mass at the identity.

Let $G$ be an infinite discrete group with identity $e$, and let $s \rightarrow L_s$ (resp. $R_s$) denote the left (resp. right) regular representation of $G$ on $l^2(G)$. For $s \in G$, let $U_s = L_s R_s$, so $U_s \xi(t) = \xi(s^{-1}ts)$ for $\xi \in H$. Write $C^*(L_G, R_G)$ for the C*-algebra generated by the unitaries $L_s, R_s$ $(s \in G)$, and $C^*(U_G)$ for the C*-subalgebra of $C^*(L_G, R_G)$ generated by the unitaries $U_s$. Let $S$ denote the characteristic function of $\{e\}$, and $P_\delta$ the projection on the one-dimensional subspace of $H$ spanned by $\delta$. In [2], using computations from [1], C. A. Akemann and P. A. Ostrand proved that $C^*(L_G, R_G)$ contains the compact operators when $G$ is the free group on two generators by showing that in this case one has $P_\delta \in C^*(U_G)$. Our theorem below provides an easier proof (and a generalization) of this result.

Following E. G. Effros [3], we say that $G$ is inner amenable if there is a state $m$ on the C*-algebra $l^\infty(G)$ such that $m(\delta) = 0$ and $m$ is invariant under the automorphisms $T_s$ $(s \in G)$ of $l^\infty(G)$ defined by $(T_s f)(t) = f(s^{-1}ts)$. Such an $m$ is called a nontrivial inner mean on $G$. Inner amenability is a considerably weaker condition on $G$ than amenability in the usual sense. The free group on two generators is an easily accessible example of a group which is not inner amenable (see [3]). Inner amenability and the behavior of $C^*(U_G)$ are related by the following theorem.

Theorem. The group $G$ is inner amenable if and only if $P_\delta \in C^*(U_G)$.

Proof. First suppose that $P_\delta \notin C^*(U_G)$. Since $U_s \delta = \delta$ for each $s \in G$, it follows that $P_\delta X = XP_\delta = (X \delta, \delta)P_\delta$ for every $X \in C^*(U_G)$, so $C^*(U_G) + CP_\delta$ is a *-algebra. We claim that it is norm-closed, and hence a C*-algebra. [Suppose $X_n + z_n P_\delta \rightarrow Y$ in norm, with $X_n \in C^*(U_G)$ and $z_n \in C$. The sequence $\{z_n\}$ must be bounded (for otherwise we could pass to a subsequence and assume $|z_n| \rightarrow \infty$, forcing $z_n^{-1}X_n + P_\delta \rightarrow 0$ and thereby contradicting $P_\delta \notin C^*(U_G)$), so we may assume that $z_n \rightarrow z \in C$ and hence...]

Received by the editors September 16, 1977 and, in revised form, November 29, 1977.

AMS (MOS) subject classifications (1970). Primary 46L05.

Key words and phrases. C*-algebra, discrete group, inner amenable.

1Research partially supported by a grant from the National Science Foundation.

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We conclude that \( Y \in C^*(U_G) + CP_\delta \). Our hypothesis permits us to define a linear functional \( \phi \) on \( C^*(U_G) + CP_\delta \) by

\[
\phi(X + zP_\delta) = (X\delta, \delta) > 0,
\]

so \( \phi \) is a state. Extend \( \phi \) to a state \( \psi \) on the algebra \( B(H) \) of bounded operators on \( H \). Since \( \psi(U_s) = 1 \), we see that \( I - U_s \) belongs to the left and right kernels of \( \psi \) for every \( s \in G \) and hence \( \psi(U_sXU_s^*) = \psi(X) \) for every \( X \in B(H) \). For \( f \in L^\infty(G) \), let \( \pi(f) \) be the corresponding multiplication operator on \( H \). Since \( U_s\pi(f)U_s^* = \pi(T_sf) \) and \( \pi(\delta) = P_\delta \), the state \( \psi \circ \pi \) on \( L^\infty(G) \) is a nontrivial inner mean for \( G \), as required.

For the converse direction, suppose that \( G \) is inner amenable. The "convergence to invariance" argument in [3] yields a net \( \{\xi_s\} \) of unit vectors in \( H \) with \( \xi_s(e) = 0 \) for every \( e \) and

\[
\lim_{s \to \infty} \|U_s\xi_s - \xi_s\| = 0
\]

for every \( s \in G \). Let \( \psi \) be any \( w^* \)-limit state of the net of vector states on \( B(H) \) corresponding to the net \( \{\xi_s\} \). We have \( \psi(U_s) = 1 \ (s \in G) \) and \( \psi(P_\delta) = 0 \). By applying \( \psi \) and the vector state corresponding to \( \delta \), one sees that the norm of the difference of \( P_\delta \) and any finite linear combination of the \( U_s \)'s must be at least \( 1/2 \), so \( P_\delta \in C^*(U_G) \).

REFERENCES

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