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MARKOV PARTITIONS ARE NOT SMOOTH[1]

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ABSTRACT. The boundaries of the sets in a Markov partition for linear Anosov diffeomorphisms of T^3 cannot be smooth.

It will be shown that the boundaries of the rectangles in a Markov partition for an Anosov automorphism of the 3-torus are never smooth. The proof constructs a certain one-dimensional invariant set and then appeals to the result of Franks [2] that such a set cannot contain a smooth arc. The existence of such an invariant set was asked by Smale (see Hirsch [3]); S. G. Hancock has also constructed such a set [7].

Let A be a 3×3 integral matrix with eigenvalues $\lambda_1, \lambda_2, \lambda_3$ such that

$$\det A = \lambda_1\lambda_2\lambda_3 = \pm 1 \text{ and each } |\lambda_i| \neq 1.$$

For definiteness we suppose $|\lambda_1| < 1$, $|\lambda_2|, |\lambda_3| > 1$. Let V^s be the line in R^3 spanned by the eigenvector for λ_1 and V^u the invariant plane corresponding to λ_2 and λ_3; so $R^3 = V^s \oplus V^u$. A compact set of the form $R = A \oplus B$ is called a *rectangle* provided $R = \overline{\text{int}\, R}$. One writes $\partial^s R = A \oplus \partial B$ and $\partial^u R = \partial A \oplus B$. A rectangle in T^3 is the projection of a rectangle in R^3 of small diameter.

The matrix A induces a diffeomorphism f on $T^s = R^3/Z^3$ by $f(x + Z^3) = Ax + Z^3$. A *Markov partition* for f is a family $\mathcal{C} = \{R_1, \dots, R_m\}$ of small connected rectangles in T^3 covering T^3 and satisfying

(i) $\text{int}(R_i \cap R_j) = \varnothing$ for $i \neq j$;
(ii) $f(\partial^s\mathcal{C}) \subset \partial^s\mathcal{C}$ where $\partial^s\mathcal{C} = \bigcup_{i=1}^m \partial^s R_i$;
(iii) $f^{-1}(\partial^u\mathcal{C}) \subset \partial^u\mathcal{C}$ where $\partial^u\mathcal{C} = \bigcup_{i=1}^m \partial^u R_i$.

The conditions $|\lambda_i| \neq 1$ means that f is an Anosov diffeomorphism and hence has Markov partitions (Sinai [6]).

PROPOSITION. $X = \bigcap_{k=0}^\infty f^k(\partial^s\mathcal{C})$ *is a compact one-dimensional invariant set for f.*

PROOF. That $f(X) = X$ follows from (ii) above. Because $X \subset \partial^s\mathcal{C}$ is nowhere dense in T^3, $\dim X \leqslant 2$ and then $\dim X \leqslant 1$ by a result of Hirsch and Williams [3].

Next one finds a periodic point $p \in \partial^s\mathcal{C}$. This is accomplished by counting the number $N_n(f)$ of fixed points of f^n in two different ways. For convenience

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assume $\lambda_1, \lambda_2, \lambda_3$ are all positive. From algebraic topology

$$N_n(f) = |\det(I - A^n)| = |-\lambda_1^n - \lambda_2^n - \lambda_3^n + (\lambda_1\lambda_2)^n + (\lambda_1\lambda_3)^n + (\lambda_2\lambda_3)^n|.$$

On the other hand one estimates $N_n(f)$ using symbolic dynamics (Manning [5] has an exact count). There is a subshift of finite type $\sigma: \Sigma_B \to \Sigma_B$ and an at most c-to-1 ($c < \infty$) surjection [1] $\pi: \Sigma_B \to T^3$ so that $\pi \circ \sigma = f \circ \pi$. The map π is 1-1 over points $x \in T^3 \setminus \bigcup_{k\in Z} f^k(\partial^s\mathcal{C} \cup \partial^u\mathcal{C})$. It follows that

$$|N_n(f) - N_n(\sigma|\Sigma_B)| \leq c \cdot \operatorname{card}\{x \in \partial^s\mathcal{C} \cup \partial^u\mathcal{C}: f^n x = x\}.$$

The local unstable manifold $W_\varepsilon^u(x)$ is defined by

$$W_\varepsilon^u(x) = x + \{v \in V^u: \|v\| < \varepsilon\}.$$

Because $s = 1$ and R_k is connected, each

$$\partial^u R_k = (W_\varepsilon^u(x_{1,k}) \cap R_k) \cup (W_\varepsilon^u(x_{2,k}) \cap R_k)$$

for certain $x_{1,k}, x_{2,k} \in R_k$. The Markov condition $f^{-1}(\partial^u\mathcal{C}) \subset \partial^u\mathcal{C}$ implies that f^{-1} maps each of the sets $\{W^u(x_{j,k}, R_k)\}_{j,k}$ into another one, by a contraction. It follows that f^{-1} (and hence f) has only finitely many periodic points in $\partial^u\mathcal{C}$.

We shall prove that $\partial^s\mathcal{C}$ contains infinitely many periodic points. Otherwise $|N_n(f) - N_n(\sigma|\Sigma_B)|$ is bounded. Now

$$N_n(\sigma|\Sigma_B) = \operatorname{Tr}(B^n) = \mu_1^n + \cdots + \mu_r^n$$

where $\mu_1, \ldots, \mu_r$ are the eigenvalues of B. Since $|\lambda_1|$, $|\lambda_1\lambda_2| = |\lambda_3|^{-1}$ and $|\lambda_1\lambda_3| = |\lambda_2|^{-1}$ are all less than 1, one would have (recall $\lambda_2\lambda_3 > 0$ for convenience) that

$$c_n = \mu_1^n + \cdots + \mu_r^n + \lambda_2^n + \lambda_3^n - (\lambda_2\lambda_3)^n$$

is bounded. Now the entropy of f equals

$$h(f) = \log(\lambda_2\lambda_3) \quad (\text{see } [\mathbf{8}]);$$

also $h(f) = h(\sigma|\Sigma_B) = \log \mu$ where μ is the largest positive eigenvalue of B, say $\mu = \mu_r$. Then $\lambda_2\lambda_3 = \mu_r$ and so

$$c_n = \mu_1^n + \cdots + \mu_{r-1}^n + \lambda_2^n + \lambda_3^n.$$

The function

$$\zeta(z) = \exp \sum_{n=1}^{\infty} \frac{c_n z^n}{n} = \frac{1}{(1 - \lambda_2 z)(1 - \lambda_3 z)(1 - \mu_1 z) \cdots (1 - \mu_{r-1} z)}$$

is analytic on $|z| < 1$ (c_n bounded) but has a pole at $z = \lambda_2^{-1}$, a contradiction.

Now let $p \in \partial^s\mathcal{C}$ with $f^n p = p$. In [1] certain finite strings of integers $J_s(x)^*$ and $J_u(x)^*$ were given which described the position of x relative to $\mathcal{C}$ and its boundaries. These strings of integers are constant over the minimal set $\{p, f(p), \ldots, f^{n-1}(p)\}$ [1, p. 916]. From this it follows that (see [1, pp. 910–912]) there are neighborhoods U_k of $f^k p$ so that

$$y \in W_\varepsilon^u(f^k p) \cap \partial^s\mathcal{C} \cap U_k \Rightarrow f^{-1}y \in \partial^s\mathcal{C}.$$

Since $f^{-1}W_\varepsilon^u(f^k p) \subset W_\varepsilon^u(f^{k-1}p)$, it follows that there is a neighborhood

$U \subset U_0$ of p so that $f^{-i}y \in W^u_\varepsilon(f^{-i}p) \cap \partial^s\mathcal{C} \cap U_{-i}$ for every $i \geqslant 0$ when $y \in W^u_\varepsilon(p) \cap \partial^s\mathcal{C} \cap U$. Hence $W^u_\varepsilon(p) \cap \partial^s\mathcal{C} \cap U \subset \bigcap_{i=0}^{\infty} f^i(\partial^s\mathcal{C}) = X$. Now $W^u_\varepsilon(p) \cap U$ contains a small 2-disk D about p in $W^u_\varepsilon(p)$. The sets $D \cap R_k$ $(k = 1, \ldots, m)$ generate a partition of D into closed sets with dense interior intersecting only in their boundaries and whose boundaries lie in $\partial^s\mathcal{C} \cap W^u_\varepsilon(p)$. There are at least two sets since $p \in \partial^s\mathcal{C}$; thus $\partial^s\mathcal{C} \cap W^u_\varepsilon(p)$ has dimension 1 since it separates open sets in R^2 [4, p. 46]. Hence $\dim X = 1$. □

THEOREM. *$\mathcal{C}$ is not a smooth partition.*

PROOF. Smoothness here means that the boundaries of the R_k's are piecewise smooth submanifolds (with corners). If $\mathcal{C}$ were smooth, then $\partial^s\mathcal{C} \cap W^u_\varepsilon(p)$ in the above proof would piecewise be a smooth 1-manifold. But a theorem of Franks [2] says that a one-dimensional invariant subset for f cannot contain a C^1 arc. □

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