PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 71, Number 1, August 1978

AN INEQUALITY FOR GENERALIZED QUADRANGLES

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ABSTRACT. Let $\mathcal{S}$ be a generalized quadrangle of order (s, t). Let X and Y be disjoint sets of pairwise noncollinear points of $\mathcal{S}$ such that each point of X is collinear with each point of Y. If $m = |X|$ and $n = |Y|$, then $(m - 1)(n - 1) \leqslant s^2$. When equality holds, severe restrictions are placed on m, n, s, and t.

I. Prolegomena. A generalized quadrangle of order (s, t), $s \geqslant 1$, $t \geqslant 1$, is a point-line incidence geometry $\mathcal{S} = (\mathcal{P}, \mathcal{L}, I)$ with point set $\mathcal{P}$, line set $\mathcal{L}$, and symmetric point-line incidence relation I satisfying the following axioms:

A1. No two points are incident with two lines in common.

A2. If x is a point not incident with a line L, then there is a unique point y incident with L and collinear with x.

A3. Each line (respectively, point) is incident with $1 + s$ points (respectively, $1 + t$ lines).

Throughout this note $\mathcal{S} = (\mathcal{P}, \mathcal{L}, I)$ will denote a generalized quadrangle (GQ) of order (s, t), $s \geqslant 1$, $t \geqslant 1$. Let $X = \{x_1, \ldots, x_m\}$ and $Y = \{y_1, \ldots, y_n\}$ be disjoint sets of pairwise noncollinear points of $\mathcal{S}$, $m \geqslant 2$ and $n \geqslant 2$. Let k_i be the number of x_j's with which y_i is collinear, $1 \leqslant i \leqslant n$, $0 \leqslant k_i \leqslant m$. Our main results consist of the following two theorems.

THEOREM I.1.

$$(1 + s) \cdot \sum_{i=1}^{n} k_i \leqslant mn + \sqrt{m^2n^2 + (s^2 - 1)(m + n)mn + (s^2 - 1)^2 mn}.$$

THEOREM I.2. *Let $k_i = m$ for all i, i.e. each y_i is collinear with each x_j. Then $(m - 1)(n - 1) \leqslant s^2$. If equality holds, then one of the following must occur.*

(i) *$m = n = 1 + s$, and each point of $Z = \mathcal{P} \setminus (X \cup Y)$ is collinear with precisely two points of $X \cup Y$.*

(ii) *$m \neq n$. If $m < n$, then $s|t$, $s < t$, $n = 1 + t$, $m = 1 + s^2/t$, and each point of $\mathcal{S}$ is collinear with either 1 or $1 + t/s$ points of Y according as it is or is not collinear with some point of X. Note: $(m - 1)|s$.*

There are two corollaries that deserve mention.

COROLLARY I.3. *If there is a GQ $\mathcal{S}$ with order (s, t), $s > 1$, then $t \leqslant s^2$. If $t = s^2$, then each triad of points has exactly $1 + s$ centers.*

Received by the editors February 14, 1977.

AMS (MOS) subject classifications (1970). Primary 05B25; Secondary 15A18.

Key words and phrases. Generalized quadrangles, Rayleigh quotient, incidence matrix.

PROOF. The inequality $t \leqslant s^2$ is due to D. G. Higman ([3], [4]). Alternate treatments appear in Bose [1] and Cameron [2]. In the present setting a proof is obtained by putting $X = \{x_1, x_2\}$ where x_1 and x_2 are not collinear, $Y = \mathrm{tr}(X) =$ the set of $1 + t$ points collinear with both x_1 and x_2, and then applying Theorem I.2. $\square$

COROLLARY I.4. *Let x and y be noncollinear points of $\mathcal{S}$ with $s > 1$ and $|\mathrm{sp}(x, y)| = 1 + p$. Then $pt \leqslant s^2$. If $pt = s^2$ and $p < t$, then each point z collinear with no point of $\mathrm{sp}(x, y)$ must be collinear with exactly $1 + t/s$ points of $\mathrm{tr}(x, y)$.*

PROOF. For the original proof and an explanation of the notation see Thas [7]. In the present setting put $X = \mathrm{sp}(x, y)$, $Y = \mathrm{tr}(x, y)$. $\square$

The proofs depend on a general matrix theoretic approach due to Sims. As the treatment in [5] does not include the "case of equality," we first give an exposition of this method.

II. A matrix-theoretic technique. If $\bar{x} = (x_1, \ldots, x_n)^T$ and $\bar{y} = (y_1, \ldots, y_n)^T$ are column vectors of real numbers, then $\bar{x} \cdot \bar{y} = \sum x_i y_i$ denotes their usual dot product. If A is a real, symmetric, $n \times n$ matrix, then for each $\bar{x} \neq \bar{0}$ define the Rayleigh quotient $R(\bar{x})$ for A by

$$R(\bar{x}) = \frac{\bar{x} \cdot A\bar{x}}{\bar{x} \cdot \bar{x}}. \tag{1}$$

It is well known that A has real characteristic roots, say $\mu_1 \leqslant \cdots \leqslant \mu_n$, and that

$$\mu_1 = \min_{\bar{x}:\, \bar{x} \neq \bar{0}} R(\bar{x}) \leqslant \max_{\bar{x}:\, \bar{x} \neq \bar{0}} R(\bar{x}) = \mu_n. \tag{2}$$

Perhaps not so well known is the following.

II.1. *Let $\bar{x}$ be a nonzero vector in R^n for which $R(\bar{x}) = \mu_i$ for either $i = 1$ or $i = n$. Then $\bar{x}$ is a characteristic vector of A belonging to the characteristic value μ_i.*

PROOF. Let $\bar{x}_1, \ldots, \bar{x}_n$ be an orthonormal basis of characteristic vectors of A ordered so that $A\bar{x}_i = \mu_i \bar{x}_i$. Let $\bar{x}$ be an arbitrary nonzero vector of R^n normalized so that $\bar{x} \cdot \bar{x} = 1$. Then $R(\bar{x}) = \bar{x} \cdot A\bar{x}$ and $\bar{x} = \sum c_i \bar{x}_i$ with $\sum c_i^2 = 1$. Hence $\mu_1 = \mu_1 \cdot \sum c_i^2 \leqslant \sum c_i^2 \mu_i = \bar{x} \cdot A\bar{x} = R(\bar{x})$, with equality holding if and only if $\mu_i = \mu_1$ whenever $c_i \neq 0$. It follows that $R(\bar{x}) = \mu_1$ if and only if $\bar{x}$ belongs to the eigenspace associated with μ_1. The argument for μ_n is similar. $\square$

We continue to let $A = (a_{ij})$ denote an $n \times n$ real symmetric matrix. Let $\Delta = \Delta_1 + \cdots + \Delta_r$ and $\Gamma = \Gamma_1 + \cdots + \Gamma_s$ be partitions of $\{1, \ldots, n\}$. Suppose that Γ is a refinement of Δ, and write $i \subseteq j$ whenever $\Gamma_i \subseteq \Delta_j$, $1 \leqslant i \leqslant s$, $1 \leqslant j \leqslant r$. Put $\delta_i = |\Delta_i|$, $\gamma_i = |\Gamma_i|$. Let

$$\delta_{ij} = \sum_{\substack{\mu \in \Delta_i \\ \gamma \in \Delta_j}} a_{\mu\nu}; \quad \gamma_{ij} = \sum_{\substack{\mu \in \Gamma_i \\ \nu \in \Gamma_j}} a_{\mu\nu}.$$

So $\delta_{ij} = \delta_{ji}$ and $\gamma_{ij} = \gamma_{ji}$ by the symmetry of A. Define the following matrices:

$$A^\Delta = \left(\frac{\delta_{ij}}{\delta_i}\right)_{1 \leqslant i,j \leqslant r}; \qquad A^\Gamma = \left(\frac{\gamma_{ij}}{\gamma_i}\right)_{1 \leqslant i,j \leqslant s}.$$

$$\bar{A}_\Delta = \operatorname{diag}\left(\sqrt{\delta_1}, \ldots, \sqrt{\delta_r}\right); \qquad \bar{A}_\Gamma = \operatorname{diag}\left(\sqrt{\gamma_1}, \ldots, \sqrt{\gamma_s}\right).$$

$$\hat{A}_\Delta = \bar{A}_\Delta A^\Delta \left(\bar{A}_\Delta\right)^{-1} = \left[\frac{\delta_{ij}}{\sqrt{\delta_i \delta_j}}\right]_{1 \leqslant i,j \leqslant r}.$$

$$\hat{A}_\Gamma = \bar{A}_\Gamma A^\Gamma \left(\bar{A}_\Gamma\right)^{-1} = \left(\frac{\gamma_{ij}}{\sqrt{\gamma_i \gamma_j}}\right)_{1 \leqslant i,j \leqslant s}.$$

Hence $\hat{A}_\Delta$ and $\hat{A}_\Gamma$ are real symmetric matrices with real characteristic values equal to those of A^Δ and A^Γ, respectively. The smallest and largest characteristic roots of $\hat{A}_\Gamma$ and $\hat{A}_\Delta$ are the minimum and maximum, respectively, of $(\bar{x} \cdot \hat{A}_\Gamma \bar{x})/(\bar{x} \cdot \bar{x})$ and $(\bar{y} \cdot \hat{A}_\Delta \bar{y})/(\bar{y} \cdot \bar{y})$, $\bar{0} \neq \bar{x} \in R^s$, $\bar{0} \neq \bar{y} \in R^r$.

Let $0 \neq \bar{y} = (y_1, \ldots, y_r)^T \in R^r$. Then put $\bar{x} = (\ldots, x_\alpha, \ldots)^T$, where $x_\alpha = y_i \sqrt{\gamma_\alpha / \delta_i}$ whenever $\alpha \subseteq i$, $1 \leqslant \alpha \leqslant s$. Then

$$\sum_{\alpha=1}^{s} x_\alpha^2 = \sum_{i=1}^{r} \left(\sum_{\alpha \subseteq i} \left(y_i \sqrt{\gamma_\alpha / \gamma_i}\,\right)^2 \right) = \sum_{i=1}^{r} \frac{y_i^2}{\delta_i} \left(\sum_{\alpha \subseteq i} \gamma_\alpha \right) = \sum_{i=1}^{r} y_i^2,$$

implying $\bar{x} \cdot \bar{x} = \bar{y} \cdot \bar{y}$. And

$$\begin{aligned}
\bar{x} \cdot \hat{A}_\Gamma \bar{x} &= \sum_{\alpha,\beta=1}^{s} x_\alpha \frac{\gamma_{\alpha\beta}}{\sqrt{\gamma_\alpha \gamma_\beta}} x_\beta \\
&= \sum_{i,j=1}^{r} \left[\sum_{\substack{\alpha \subseteq i \\ \beta \subseteq j}} \frac{\gamma_{\alpha\beta}}{\sqrt{\gamma_\alpha \gamma_\beta}} \cdot \frac{y_i \sqrt{\gamma_\alpha}}{\sqrt{\delta_i}} \cdot \frac{y_j \sqrt{\gamma_\beta}}{\sqrt{\delta_j}} \right] \\
&= \sum_{i,j=1}^{r} y_i \left[\sum_{\substack{\alpha \subseteq i \\ \beta \subseteq j}} \frac{\gamma_{\alpha\beta}}{\sqrt{\delta_i \delta_j}} \right] y_j \\
&= \sum_{i,j=1}^{r} y_i \left[\frac{\delta_{ij}}{\sqrt{\delta_i \delta_j}} \right] y_j = \bar{y} \cdot \hat{A}_\Delta \bar{y}.
\end{aligned}$$

This implies that any value of $(\bar{y} \cdot \hat{A}_\Delta \bar{y})/(\bar{y} \cdot \bar{y})$ is also a value of $(\bar{x} \cdot \hat{A}_\Gamma \bar{x})/(\bar{x} \cdot \bar{x})$. Hence the following is a corollary of (2) and II.1.

II.2. *If $\mu_1 \leqslant \cdots \leqslant \mu_r$ are the characteristic roots of A^Δ and $\lambda_1 \leqslant \cdots \leqslant \lambda_s$ are the characteristic roots of A^Γ, then $\lambda_1 \leqslant \mu_1 \leqslant \mu_r \leqslant \lambda_s$. If $\bar{y} = (y_1, \ldots, y_r)^T$ satisfies $A^\Delta \bar{y} = \lambda_1 y$ (so $\lambda_1 = \mu_1$), then $A^\Gamma \bar{x} = \lambda_1 \bar{x}$, where $\bar{x} = (\ldots, x_\alpha, \ldots)^T$ is defined by $x_\alpha = y_i$ whenever $\alpha \subseteq i$. A similar result holds in case $\lambda_n = \mu_n$.*

PROOF. The first part of the result is evident. So let $\bar{0} \neq \bar{y} = (y_1, \ldots, y_r)^T$ satisfy $A^\Delta \bar{y} = \lambda_1 \bar{y} = \mu_1 \bar{y}$. Then $\bar{A}_\Delta \bar{y} = (y_1\sqrt{\delta_1}, \ldots, y_r\sqrt{\delta_r})^T$ is a characteristic vector of $\hat{A}_\Delta$ belonging to $\lambda_1 = \mu_1$. Hence $\bar{z} = (\ldots, z_\alpha, \ldots)^T$, $z_\alpha = y_i\sqrt{\gamma_\alpha}$ for $\alpha \subseteq i$, is a characteristic vector of $\hat{A}_\Gamma$ belonging to λ_1 (by the proof of II.1). It follows that $\bar{x}$ as given in the statement of II.2 is a characteristic vector of A^Γ associated with λ_1. A similar proof holds in case $\lambda_n = \mu_n$. $\square$

III. Applications to generalized quadrangles. Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, I)$ be a GQ of order (s, t). Let X and Y be as in the hypothesis of Theorem I.1, and put $Z = \mathcal{P} \setminus (X \cup Y)$, so $|Z| = r = v - (m + n)$, where $v = (1 + s)(1 + st) = |\mathcal{P}|$. For some ordering of $\mathcal{P}$ let A be the (0, 1)-matrix $A = (a_{ij})$ defined by $a_{ij} = 1$ if the ith and jth points of $\mathcal{P}$ are *not* collinear in $\mathcal{S}$; $a_{ij} = 0$ otherwise. It follows that A is symmetric with minimum polynomial given by $f(x) = (x + s)(x - t)(x - ts^2)$. Let $\Delta = \Delta_1 + \Delta_2 + \Delta_3$ be the partition of $\{1, \ldots, v\}$ determined by X, Y, and Z; i.e. points of X, Y, Z, respectively, are indexed by Δ_1, Δ_2, Δ_3, respectively. As $\delta_i = |\Delta_i|$, we have $\delta_1 = m$, $\delta_2 = n$, $\delta_3 = v - (m + n)$, $\delta_{11} = n(n - 1)$, $\delta_{12} = \delta_{21} = \Sigma_{i=1}^n (m - k_i)mn = \Sigma$, where $\Sigma = \Sigma_{i=1}^n k_i$. Since $\Sigma_{j=1}^3 (\delta_{ij}/\delta_i) = ts^2$, we also have $\delta_{13} = \delta_1 ts^2 - \delta_{12} - \delta_{11} = ts^2 m - (mn - \Sigma) - m(m - 1)$. Similarly, $\delta_{23} = ts^2 n - (mn - \Sigma) - n(n - 1)$. Using these results it is now routine to complete the calculation of A^Δ.

$$A^\Delta = \begin{pmatrix} m - 1 & n - \Sigma/m & ts^2 + 1 - m - n + \Sigma/m \\ m - \Sigma/n & n - 1 & ts^2 + 1 - m - n + \Sigma/n \\ A_1 & A_2 & A_3 \end{pmatrix}$$

where

$$A_1 = \frac{m[ts^2 + 1 - m - n] + \Sigma}{v - m - n}, \quad A_2 = \frac{n[ts^2 + 1 - m - n] + \Sigma}{v - m - n}$$

and

$$A_3 = ts^2 - \frac{(m + n)[ts^2 + 1 - m - n] + 2\Sigma}{v - m - n}.$$

Let $(x - ts^2)(x - r_1)(x - r_2)$ be the characteristic polynomial of A^Δ with the roots ordered so that $r_1 \leqslant r_2 \leqslant ts^2$. Let $\Gamma = \Gamma_1 + \cdots + \Gamma_v$ be the identity partition of $\{1, \ldots, v\}$, so Γ is a refinement of Δ. Then $A^\Gamma = A$ has numerical range $[-s, ts^2]$ which must then contain all characteristic roots of A^Δ. Indeed, the proof of Theorem I.1 amounts to calculating r_1 and using the inequality $-s \leqslant r_1$. We now proceed to do this.

Put $(x - r_1)(x - r_2) = x^2 - bx + c$, so that $2r_1 = b - \sqrt{b^2 - rc}$. Hence $-s \leqslant r_1$ simplifies to

$$0 \leqslant s^2 + bs + c, \quad b = r_1 + r_2 = \operatorname{tr}(A^\Delta) - ts^2, \quad c = \det(A^\Delta)/ts^2. \tag{4}$$

It is easy to calculate $\operatorname{tr}(A^\Delta)$ from (3) and then to write b as follows.

$$b = \frac{(m + n)(s + st + 2) - 2v - 2\Sigma}{v - m - n}. \tag{5}$$

To calculate $\det(A^\Delta)$, add the first and second columns of A^Δ to the third column and then subtract the first row from the second. At this point $\det(A^\Delta)$ appears as follows.

$$\det(A^\Delta) = ts^2 \begin{vmatrix} m - 1 & n - \Sigma/m & 1 \\ 1 - \Sigma/n & \Sigma/m - 1 & 0 \\ \dfrac{m[ts^2 + 1 - m - n] + \Sigma}{v - m - n} & \dfrac{n[ts^2 + 1 - m - n] + \Sigma}{v - m - n} & 1 \end{vmatrix}. \tag{6}$$

Expanding by the third column and simplifying, one may calculate c to be as follows.

$$c = \det(A^\Delta)/ts^2 = \frac{(1 + s + st)(2\Sigma - m - n) + v - v\Sigma^2/mn}{v - m - n}. \tag{7}$$

Using the values for b and c given in (5) and (7), (4) may be rewritten as follows.

$$0 \leqslant (s - 1)(m + n + s^2 - 1)mn + 2mn\Sigma - (1 + s)\Sigma^2. \tag{8}$$

Equality in (8) gives two roots Σ_1 and Σ_2 for which (8) says $\Sigma_1 \leqslant \Sigma \leqslant \Sigma_2$, if $\Sigma_1 \leqslant \Sigma_2$. But Σ_2 is easily evaluated.

$$\Sigma_2 = \frac{mn + \sqrt{m^2n^2 + (s^2 - 1)(m + n)mn + (s^2 - 1)^2 mn}}{1 + s}. \tag{9}$$

Clearly $\Sigma \leqslant \Sigma_2$ is just the inequality in Theorem I.1. If each $k_i = m$, then $\Sigma = mn$, and the inequality of Theorem I.1 reduces to $(m - 1)(m - 1) \leqslant s^2$, the inequality of Theorem I.2.

We now use II.2 to investigate the case of equality in Theorem I.2. Suppose that $k_i = m$ for all i, so $\Sigma = mn$, and suppose that $(m - 1)(n - 1) = s^2$, so $-s$ is a characteristic root of A^Δ. Hence a nonzero characteristic vector of A^Δ belonging to $-s$ must span the null space of $A^\Delta + sI$.

$$A^\Delta + sI = \begin{bmatrix} m - 1 + s & 0 & ts^2 + 1 - m \\ 0 & n - 1 + s & ts^2 + 1 - n \\ * & * & * \end{bmatrix}, \tag{10}$$

where we need not bother to calculate the third row, since the rank must equal 2. Clearly $\bar{y} = (y_1, y_2, 1)^T$ spans the null space of $A^\Delta + sI$, where

$$y_1 = \frac{m - 1 - ts^2}{s + m - 1}; \qquad y_2 = \frac{n - 1 - ts^2}{s + n - 1}. \tag{11}$$

Let us assume that the points of $\mathcal{P}$ are ordered (for the construction of A) so that the first m points are those of X, the next n points are those of Y, and the last $v - m - n$ points are those of Z. Then by II.2, $\bar{x}$ must be a characteristic vector of $A^\Gamma = A$ belonging to $\lambda_1 = -s$, where $\bar{x}$ is as follows.

$$\bar{x} = \Big(\underbrace{y_1, \ldots, y_1}_{m \text{ times}},\ \underbrace{y_2, \ldots, y_2}_{n \text{ times}},\ \underbrace{1, \ldots, 1}_{v-m-n \text{ times}}\Big)^T. \tag{12}$$

For the first $m + n$ rows of A this yields no new information. But let $z \in Z$ be the ith point, $i > m + n$. Suppose z is not collinear with t_1 points of X, is not collinear with t_2 points of Y, and hence is not collinear with $ts^2 - t_1 - t_2$ points of Z. Then the product of the ith row of A with $\bar{x}$, which must equal $-s$, is actually $t_1 y_1 + t_2 y_2 + ts^2 - t_1 - t_2 = s$. After a little simplification this becomes

$$\frac{t_1}{s + m - 1} + \frac{t_2}{s + n - 1} = 1. \tag{13}$$

If z lies on a line joining a point of X and a point of Y, then $t_1 = m - 1$ and $t_2 = n - 1$, i.e., since $\mathcal{S}$ has no triangles, z is collinear with a unique point of X and with a unique point of Y. On the other hand, if z is not on such a line either $t_1 = m$ or $t_2 = n$. Suppose $t_1 = m$, so z is collinear with no point X. Using (13) we find that the number of points of Y collinear with z is

$$n - t_2 = 1 + (n - 1)/s. \tag{14}$$

Similarly, any point of $\mathcal{P}$ collinear with no point of Y must be collinear with $1 + (m - 1)/s$ points of X. If $m = n = s + 1$, this says each point not on a line joining a point of X with a point of Y must be collinear with two points of X and none of Y or with two of Y and none of X. If $1 < m < s + 1$, so $1 + (m - 1)/s$ is not an integer, then each point of $\mathcal{P}$ is collinear with some point of Y. This implies that each point z of Z is either on a line joining points of X and Y or is collinear with $1 + (n - 1)/s \geqslant 3$ points of Y. Clearly $n \leqslant 1 + t$. Suppose $n < 1 + t$ and let $x_1 \in X$. Then there is some line L through x_1 not incident with any point of Y. But then any point z on L, $z \neq x_1$, cannot be collinear with any point of Y, a contradiction. Hence it must be that $n = 1 + t$, from which it follows that $m = 1 + s^2/t$. This essentially completes the proof of Theorem I.2.

A similar treatment is available for the restriction on the parameters of a subquadrangle, a combinatorial proof of which is found in [**6**].

References

1. R. C. Bose, *Geometric and pseudo-geometric graphs* $(q^2 + 1, q + 1, 1)$, J. Geometry **2** (1972), 75–93.

2. P. J. Cameron, *Partial quadrangles*, Quart. J. Math. Oxford Ser. **25** (1974), 1–13.

3. D. G. Higman, *Partial geometries, generalized quadrangles, and strongly regular graphs*, Atti Convegno di Geometria e sue Applicazioni, Perugia, 1971.

4. ______, *Invariant relations, coherent configurations, and generalized polygons*, Combinatorics, Part 3, Math. Centre Tracts #57, Amsterdam, 1974, 27–43.

5. S. E. Payne, *Finite generalized quadrangles: a survey*, Proc. Internat. Conf. Proj. Planes, Washington State Univ. Press, 1973, 219–261.

6. J. A. Thas, *A remark concerning the restriction on the parameters of a 4-gonal subconfiguration*, Simon Stevin **48** (1974–75), 65–68.

7. ______, *On generalized quadrangles with parameters $s = q^2$ and $t = q^3$*, Geometria Dedicata **5** (1976), 485–496.

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