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A REFINEMENT OF THE ARITHMETIC MEAN-GEOMETRIC MEAN INEQUALITY

D. I. CARTWRIGHT AND M. J. FIELD

ABSTRACT. Upper and lower bounds are given for the difference between the arithmetic and geometric means of n positive real numbers in terms of the variance of these numbers.

In this note we prove a simple refinement of the arithmetic mean-geometric mean inequality. Our result solves a problem posed by Kenneth S. Williams in [5] and generalizes an inequality on p. 215 of [3]. Other estimates for the difference between the means are discussed in [2], [3] and [4].

THEOREM. *Suppose that* $x_k \in [a, b]$ *and* $p_k \geqslant 0$ *for* $k = 1, \ldots, n$, *where* $a > 0$, *and suppose that* $\sum_{k=1}^n p_k = 1$. *Then, writing* $\bar{x} = \sum_{k=1}^n p_k x_k$, *we have*

$$\frac{1}{2b} \sum_{k=1}^{n} p_k(x_k - \bar{x})^2 \leqslant \bar{x} - \prod_{k=1}^{n} (x_k^{p_k}) \leqslant \frac{1}{2a} \sum_{k=1}^{n} p_k(x_k - \bar{x})^2. \quad (1)$$

In particular, if $p_k = 1/n$ *for each* k, *then*

$$\frac{1}{2bn^2} \sum_{j<k} (x_j - x_k)^2 \leqslant \frac{x_1 + \cdots + x_n}{n} - \left(\prod_1^n x_j\right)^{1/n} \leqslant \frac{1}{2an^2} \sum_{j<k} (x_j - x_k)^2.$$

REMARK. These inequalities may be generalized as follows: Let m be a probability measure on $[a, b]$, where $a > 0$, and let $\mu = \int_a^b t\, dm(t)$ and $\sigma^2 = \int_a^b (t - \mu)^2\, dm(t)$ be the mean and variance of m. Then

$$\frac{1}{2b}\sigma^2 \leqslant \mu - \exp\left(\int_a^b \log(t)\, dm(t)\right) \leqslant \frac{1}{2a}\sigma^2.$$

This follows from our theorem and the weak* density of the measures of the form $\sum_{k=1}^n p_k \delta_{x_k}$ (where δ_x denotes the probability measure which is concentrated at the point x) in the set of all probability measures on $[a, b]$. (See [1, p. 709].) Notice that the inequality

$$\exp\left(\int_a^b \log(t)\, dm(t)\right) \leqslant \mu$$

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is just a special case of Jensen's inequality.

LEMMA. *Let* $0 \leqslant q \leqslant 1$. *Then for all* $t \geqslant 0$ *we have*

$$1 + qt + \frac{q(q-1)}{2}\, t^2 \leqslant (1+t)^q \leqslant 1 + qt + \frac{q(q-1)}{2}\, \frac{t^2}{1+t}.$$

PROOF. After a little algebra we see that

$$\begin{aligned} \frac{d}{dt} \log\left(1 + qt + \frac{q(q-1)}{2}\, \frac{t^2}{1+t}\right) &= \frac{q}{1+t}\left\{\frac{2 + (2+2q)t + (1+q)t^2}{2 + (2+2q)t + q(1+q)t^2}\right\} \\ &\geqslant \frac{q}{1+t} \quad \text{since } 0 \leqslant q \leqslant 1 \\ &= \frac{d}{dt}\log(1+t)^q. \end{aligned}$$

Since $(1+t)^q$ and $1 + qt + (q(q-1)/2)(t^2/(1+t))$ agree at $t = 0$, the right-hand inequality is proved.

The left-hand inequality may be proved in the same way, or by using the Taylor expansion of $(1+t)^q$.

PROOF OF THE THEOREM. The inequalities (1) are trivially valid if $n = 1$. Let $n = 2$. We may suppose that $x_2 \geqslant x_1$. Writing $x_2 = (1+t)x_1$, with $t \geqslant 0$, and writing $p_2 = q, p_1 = 1 - q$, the desired inequalities (1) become

$$\frac{q(1-q)}{2b}\, t^2x_1^2 \leqslant x_1\{1 + qt - (1+t)^q\} \leqslant \frac{q(1-q)}{2a}\, t^2x_1^2,$$

which follows immediately from our lemma, noting that $a \leqslant x_1 \leqslant (1+t)x_1 \leqslant b$.

Suppose now that $n \geqslant 3$ and that the inequalities (1) have been proved for all admissible x_k's and p_k's with $n-1$ replacing n.

Fix $x_1, \ldots, x_n$. We may assume that the x_k's are distinct, for otherwise the inequalities follow from the induction hypothesis. Let us consider the left-hand inequality. Define

$$f(p) = f(p_1, \ldots, p_n) = \bar{x} - \prod_{k=1}^{n} (x_k^{p_k}) - \frac{1}{2b} \sum_{k=1}^{n} p_k(x_k - \bar{x})^2$$

for $p \in S = \{p = (p_1, \ldots, p_n): p_k \geqslant 0 \text{ for each } k\}$.

There is a point p° of S where f is minimized subject to the constraint $\Sigma p_k = 1$. If p° lies on the boundary of S, then some component of p° is zero, and hence $f(p^\circ) \geqslant 0$ by the induction hypothesis, and so the left-hand inequality holds.

If p° is an interior point of S, then we may use the Lagrange multiplier method to obtain a real number λ such that at p°,

$$\frac{\partial f}{\partial p_j} = \lambda \frac{\partial}{\partial p_j}\left(\sum_{k=1}^{n} p_k - 1\right) \quad \text{for all } j,$$

i.e.

$$x_j - (\log x_j)\prod_1^n (x_k^{p_k}) - \frac{(x_j - \bar{x})^2}{2b} = \lambda.$$

Thus each x_j is a solution of the equation (in ξ)

$$(1 + \bar{x}/b)\xi - \tilde{x}\log(\xi) - \xi^2/2b = \lambda + \bar{x}^2/2b \tag{2}$$

(writing $\tilde{x}$ for $\Pi(x_k^{p_k})$).

Now between any two roots of (2) there is by Rolle's theorem a root of

$$1 + \bar{x}/b - \tilde{x}/\xi - \xi/b = 0,$$

i.e. of

$$\xi^2 - (b + \bar{x})\xi + b\tilde{x} = 0. \tag{3}$$

Since (3) has at most 2 solutions, equation (2) has at most 3 solutions. The larger root of (3) is, since $\tilde{x} \leqslant \bar{x}$,

$$\left(b + \bar{x} + \sqrt{(b + \bar{x})^2 - 4b\tilde{x}}\,\right)/2 \geqslant b.$$

Hence equation (2) has at most 2 solutions in $[a, b]$. Since each x_j is a solution and since the x_j's are distinct, we must have $n \leqslant 2$, contrary to assumption.

Thus p° must be a boundary point of S, and so the left-hand inequality is proved.

The right-hand inequality may be proved in the same way by replacing b by a in the definition of f and by noting that the smaller root of the equation corresponding to (3) is $\leqslant a$.

REMARK. Examination of the above proof shows that the inequalities in (1) are strict unless the x_k's corresponding to nonzero p_k's are all equal. Furthermore, the constants $1/2a$ and $1/2b$ in (1) are the best possible. For in the case $n = 2$ we have

$$\frac{\bar{x} - \Pi(x_k^{p_k})}{\sum p_k(x_k - \bar{x})^2} = \frac{1 + qt - (1 + t)^q}{q(1 - q)t^2 x_1}$$

if $0 < q < 1$ and $t > 0$ (in the notation of the first paragraph of the proof). It is easy to see that the limit of this expression as t tends to zero is $1/2x_1$, and since $x_1 \in [a, b)$ is arbitrary, the result follows.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SYDNEY, SYDNEY, N.S.W. 2006, AUSTRALIA